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1. Introduction

This paper presents a mathematical theory for the deformations of isotropic shells under the applications of edge forces. The theory is obtained by imposing the principal of virtual work on the class of deformations which satisfy the Kirchhoff hypothesis, i.e. those deformations which carry normals to the undeformed middle surface into normals to the deformed middle surface with no change in length along normals. It is not assumed that displacements, strains, or slopes are small. Although in the derivation of the model it is not assumed that the shell is thin, we do not expect the theory to be physically realistic unless the shell is in fact thin.

Imposing the principal of virtual work yields a system of differential equations for the three components of position of the deformed middle surface and six relations between the applied edge forces and the deformed middle surface. It will be shown that the system of differential equations can be expressed as a tenth order system. The six relations between edge forces and deformed middle surface contain two arbitrary functions and hence represent four boundary constraints on the deformed middle surface when the edge forces are prescribed. It will be shown that to be physically realistic one of the arbitrary functions should be chosen to be zero. We then have five boundary constraints for the tenth order system of differential equations. Hence after prescribing the edge forces we expect the deformation to be uniquely determined unless buckling occurs.

Since it has recently been proved rigorously [1] that the nonlinear equations of the three dimensional theory reduce under suitable hypotheses to the equations of von Karman and Föppl for plates, it seems essential that any nonlinear shell theory which purports to be more general than the von Karman and Föppl theory should also reduce to that theory when one makes suitable restrictions. Unfortunately, as will be shown in this paper, the theory derived on the basis of the hypothesis stated above, does not meet this test. It is felt that it is worthwhile to point this out. In fact, Naghdi and Nordgren [2], for example, have recently presented a theory for the treatment of thin shells in which the essential hypothesis is the same as that of this paper. It is not easy to decide directly whether the the theory of Naghdi and Nordgren is equivalent to the one presented in this paper, but the author has checked the point regarded here as important, i.e. that the theory of Naghdi and Nordgren does not yield the von Karman-Föppl equations with the correct coefficients when it is formulated for thin plates. Thus it also does not agree with the von Karman-Föppl equations. The fact that others are continuing work on theories of this type increases the importance of calling attention to this defect in such theories.

2. The Three Dimensional Theory

In deriving the shell theory we attempt to copy a derivation of the complete three dimensional nonlinear theory. In this section we present that derivation of the three dimensional theory.

Consider a fixed rectangular Cartesian reference frame X. Let (x_1, x_2, x_3) be the position of a particle in the undeformed state and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be the position of the same particle in the deformed state. The Eulerian coordinates $\bar{x}_i$ are considered as functions of the Lagrangian coordinates x_i.

Let $p_{ij} = \dfrac{\partial \bar{x}_i}{\partial x_j}$ and let $W = W(p_{ij}, x_i)$ be the strain energy per unit undeformed volume. The elastic properties of the material are then determined by the choice of W.

Let $E = \iiint_R W dv$ be the total strain energy where R is the undeformed region and dv is the element of undeformed volume.

Let (t_1, t_2, t_3) be the force vector applied to the boundary $\bar{S}$ of the deformed region $\bar{R}$. The units of t_i are force per unit deformed area.

The principal of virtual work states that $\dot{E} = \iint_{\bar{S}} t_i \dot{\bar{x}}_i d\bar{S}$ where the summation convention is used, a super dot denotes the variation, and $d\bar{S}$ is the element of deformed area.

Letting $q_{ij} = \dfrac{\partial W}{\partial p_{ij}}$, we have $\dot{E} = \iiint_R q_{ij} \dfrac{\partial \dot{\bar{x}}_i}{\partial x_j} dv$. Using the divergence theorem this becomes $\dot{E} = \iint_S \dot{\bar{x}}_i q_{ij} n_j dS - \iiint_R \dot{\bar{x}}_i \dfrac{\partial q_{ij}}{\partial x_j} dv$ where S is the boundary of R, (n_1, n_2, n_3) is the unit outer normal vector to S, and dS is the element of undeformed area.

Hence the principal of virtual work yields the differential equations

$$\frac{\partial q_{ij}}{\partial x_j} = 0 \tag{2.1}$$

for the functions $\bar{x}_i$ and the relations

(2.2) $$t_i = q_{ij}n_j \frac{dS}{d\overline{S}}$$

between the force and deformation.

Letting $(\overline{n}_1, \overline{n}_2, \overline{n}_3)$ be the unit outer normal vector to $\overline{S}$, it can be shown that $q_{ij}n_j \frac{dS}{d\overline{S}} = \frac{1}{|p|} q_{ij}p_{kj}\overline{n}_k$ where $|p|$ is the determinant of the matrix $p = (p_{ij})$. Thus if we let

(2.3) $$\tau_{ij} = \frac{1}{|p|} q_{ik}p_{jk}$$

the relations between force and deformation become

(2.4) $$t_i = \tau_{ij}\overline{n}_j \; .$$

Also one can show that

(2.5) $$\frac{\partial \tau_{ij}}{\partial \overline{x}_j} = 0 \; ,$$

and

(2.6) $$\tau_{ij} = \tau_{ji}$$

the latter being true if W is invariant under rigid displacements. Thus one obtains the usual differential equations and symmetry for the components, τ_{ij}, of the stress tensor and also the usual relationship between the τ_{ij}'s and the applied forces.

For a homogeneous isotropic material we have [3]

(2.7) $$q_{ij} = W_1 p_{ij} + W_2 c_{ij} + W_3 p_{ik}c_{\ell k}p_{\ell j} \; .$$

Here the W_i's are functions of the principal strains and the c_{ij}'s are the components of the rotation matrix c. Specifically by principal strains we mean the three stationary values of

$\frac{d\bar{s}}{ds} - 1$ where ds and $d\bar{s}$ are arc length in the undeformed and deformed bodies respectively. Letting s_i be the sum of the i-th powers of the principal strains, we have $W_1 = 2\left(\frac{\partial W}{\partial s_2} - 3\frac{\partial W}{\partial s_3}\right)$, $W_2 = \frac{\partial W}{\partial s_1} - 2\frac{\partial W}{\partial s_2} + 3\frac{\partial W}{\partial s_3}$, $W_3 = 3\frac{\partial W}{\partial s_3}$. The rotation matrix c is defined by requiring that c be orthogonal, det c = 1, and c*p (or equivalently pc*) is symmetric and positive definite. The matrix c then rotates the directions of principal strain in the undeformed body into the corresponding directions of principal strain in the deformed body.

Also $c^*p = \sqrt{p^*p}$ so that the eigenvectors of c^*p have the direction of principal strain in the undeformed body and its eigenvalues are the stationary values of $\frac{d\bar{s}}{ds}$.

In terms of p and c, $s_i = [(c^*p - 1)^i]$ where the square bracket denotes the trace of the matrix.

The above expression for q_{ij} will be used in deriving our theory for isotropic shells.

3. Terminology for Shell Theory

Let $\vec{r} = \vec{r}(\theta_1, \theta_2)$ be the position vector to the undeformed middle surface, θ_1 and θ_2 being parameters for the middle surface. Let

$$(3.1) \qquad \vec{g}_\alpha = \vec{r}_{,\alpha} \quad \text{and} \quad \vec{g}_3 = \frac{\vec{g}_1 \times \vec{g}_2}{|\vec{g}_1 \times \vec{g}_2|} .$$

Here and in all that follows Greek indices take on the values 1, 2 and Latin indices take on the values 1, 2, 3. Also $,\alpha = \frac{\partial}{\partial\theta_\alpha}$ and $,i = \frac{\partial}{\partial\theta_i}$. Observe that $\vec{g}_3$ is a unit normal vector to the undeformed middle surface.

We introduce curvilinear coordinates θ_1, θ_2, θ_3 in the undeformed shell by letting

$$(3.2) \qquad \vec{x} = (x_1, x_2, x_3) = \vec{r} + \theta_3 \vec{g}_3 \ .$$

Observe that $\vec{r}$ and $\vec{g}_i$ are all independent of θ_3. Then θ_3 is the distance from (x_1, x_2, x_3) to the undeformed middle surface along a normal to the undeformed middle surface. We assume that the shell is thin enough so that normals to the undeformed middle surface do not intersect in the shell.

Let

$$(3.3) \qquad g_{ij} = \vec{g}_i \cdot \vec{g}_j \ , \quad (g^{ij}) = (g_{ij})^{-1} \ , \quad \text{and} \quad \vec{g}^i = g^{ij} \vec{g}_j \ .$$

Then the $g_{\alpha\beta}$'s and $g^{\alpha\beta}$'s are the covariant and contravariant components of the usual metric tensor for the undeformed middle surface. Also $g_{3\alpha} = g^{3\alpha} = 0$ and $g_{33} = g^{33} = 1$. Finally $g^{ij} = \vec{g}^i \cdot \vec{g}^j$, $\vec{g}_i \cdot \vec{g}^j = \delta_i^j$, and $\vec{g}_3 = \vec{g}^3$.

All vectors associated with the shell will be expressed in terms of their components with respect to the vectors $\vec{g}_i$ or $\vec{g}^i$. This is done because we will be integrating the components of vectors with respect to θ_3. Since the vectors $\vec{g}_i$ and $\vec{g}^i$ are independent of θ_3, such integration yields the components with respect to the vectors $\vec{g}_i$ or $\vec{g}^i$ of the integral of the associated vector.

We express the vectors $\vec{g}_i$ and $\vec{g}^i$ in terms of our fixed Cartesian coordinate system as follows:

$$(3.4) \qquad \vec{g}_i = (a_i^1, a_i^2, a_i^3) \ , \quad \vec{g}^i = (A_1^i, A_2^i, A_3^i) \ .$$

Then

$$(a^i_j) = (A^i_j)^{-1} , \quad g_{ij} = a^k_i a^k_j , \quad \text{and} \quad g^{ij} = A^i_k A^j_k .$$

Given any indexed set of functions associated with the Cartesian coordinates (such as the p_{ij}'s of Section 2) we associate indexed sets with the curvilinear coordinates as follows: $P_{ij} = a^m_i a^n_j p_{mn}$, $P_i{}^j = a^m_i A^j_n p_{mn}$, $P^i{}_j = A^i_m a^n_j p_{mn}$, $P^{ij} = A^i_m A^j_n p_{mn}$. We then have $P_{ij} = g_{ik} P^k{}_j$, etc., so that the quantities g^{ij} and g_{ij} can be used to raise and lower indices in the usual way. Actually this means the indices 1, 2 are raised and lowered using the quantities $g^{\alpha\beta}$ and $g_{\alpha\beta}$ and the index 3 has the same meaning regardless of whether it is up or down. This procedure for generating indexed sets associated with the curvilinear coordinates and for raising and lowering the indices applies regardless of the number of indices involved. Under changes of curvilinear coordinates of the middle surface we notice that the quantities $P_{\alpha\beta}$ transform like the components of a covariant tensor of rank two, the quantities $P_{\alpha 3}$ and $P_{3\alpha}$ like the components of a covariant tensor of rank one, and P_{33} is invariant. Similar statements hold for indexed sets with any number of indices.

In Table I we have listed in the left column certain indexed sets associated with the fixed Cartesian coordinate system and in the right hand column one of the corresponding indexed sets associated with the curvilinear coordinates.

Table I

x-components	θ-components
p_{ij}	P_{ij}
c_{ij}	C_{ij}
q_{ij}	Q_{ij}
$e_{\alpha\beta}$	$\varepsilon_{\alpha\beta}$
e_{ijk}	ε_{ijk}

Here $e_{\alpha\beta} = \pm 1$ ($e_{ijk} = \pm 1$) according as $\alpha\beta(ijk)$ is an even or odd permutation of the integers 1, 2 (1,2,3). If two indices are equal $e_{\alpha\beta} = e_{ijk} = 0$. It can then be shown that

$$(3.5)\qquad \begin{cases} \varepsilon_{\alpha\beta} = \sqrt{g}\, e_{\alpha\beta} \,, & \varepsilon^{\alpha\beta} = \dfrac{1}{\sqrt{g}}\, e_{\alpha\beta} \\[2ex] \varepsilon_{ijk} = \sqrt{g}\, e_{ijk} \,, & \varepsilon^{ijk} = \dfrac{1}{\sqrt{g}}\, e_{ijk} \end{cases}$$

where $g = \det(g_{ij})$.

In terms of this notation it can be shown that

$$(3.6)\qquad \begin{cases} \vec{g}_i \times \vec{g}_j = \varepsilon_{ijk}\, \vec{g}^{\,k} \\[2ex] \vec{g}^{\,i} \times \vec{g}^{\,j} = \varepsilon^{ijk}\, \vec{g}_k \,. \end{cases}$$

It is easily verified that

$$(3.7)\qquad Q^{ij} = \frac{\partial W}{\partial P_{ij}} \,, \qquad Q^{i}{}_{j} = \frac{\partial W}{\partial P_{i}{}^{j}} \,, \text{ etc.,}$$

and from (2.7)

$$(3.8)\qquad Q_{ij} = W_1 P_{ij} + W_2 C_{ij} + W_3 P_{ik} C_{\ell}{}^{k}\, P^{\ell}{}_{j}$$

for an isotropic material.

C_{ij} is determined by the conditions $C_{ki}C^k{}_j = g_{ij}$, $\det(C_{ij}) > 0$, and $(C_{ki}P^k{}_j)$ (or equivalently $(P_{ik}C_j{}^k)$) is symmetric and positive definite.

If $d^i\vec{g}_i$ and $D^i\vec{g}_i$ are unit vectors in corresponding directions of principal strain in the strained and unstrained bodies, then $C^{ij}D_j = d^i$.

We also observe the following. Let ℓ_{ij} be an indexed set associated with Cartesian coordinates such that $\ell_{ij} = \ell_{ji}$. Let (y_1,y_2,y_3) be an eigenvector of the matrix (ℓ_{ij}) and λ a corresponding eigenvalue. Let L_{ij} and Y_i be indexed sets associated with the curvilinear coordinates derived from ℓ_{ij} and y_i respectively. Then (L_{ij}) and (L^{ij}) are symmetric and $L_i{}^j = L^i{}_j$. We write L^i_j for $L^i{}_j$ and $L_i{}^j$. Then $L^i_jY^j = \lambda Y^i$ so that λ is an eigenvalue of L^i_j and $(y_1,y_2,y_3) = Y^i\vec{g}_i$.

It follows that if $D^i\vec{g}_i$ is a vector in a direction of principal strain in the undeformed body and λ is the corresponding value of $\frac{d\bar{s}}{ds}$, then $C^{ki}P_{kj}\,D^j = \lambda D^i$ and $P^{ki}P_{kj}\,D^j = \lambda^2D^i$.

Next we introduce

$$b_{\alpha\beta} = \vec{g}_3\cdot\vec{r}_{,\alpha\beta} \tag{3.9}$$

the components of the second fundamental form of the undeformed middle surface, and we define $\Gamma_{i\alpha}^{\,j}$ by the equations

$$\vec{g}_{i,\alpha} = \Gamma_{i\alpha}^{\,j}\,\vec{g}_j \ . \tag{3.10}$$

It can be shown that the $\Gamma_{\alpha\beta}^{\,\gamma}$'s are the usual Christoffel symbols for the metric of the undeformed middle surface, $\Gamma_{\alpha\beta}^{\,3} = b_{\alpha\beta}$, $\Gamma_{3\beta}^{\,\alpha} = -b^{\alpha}_{\beta}$, and $\Gamma_{3\alpha}^{\,3} = 0$.

Since $\vec{g}^i \cdot \vec{g}_j = \delta^i_j$, we have

$$\vec{g}^i{}_{,\alpha} \cdot \vec{g}_j = -\vec{g}^i \cdot \vec{g}_{j,\alpha} = -\vec{g}^i \cdot \Gamma^k_{j\alpha} \vec{g}_k = -\Gamma^i_{j\alpha} ,$$

and hence

$$(3.11) \qquad \vec{g}^i{}_{,\alpha} = -\Gamma^i_{j\alpha} \vec{g}^j .$$

If $u^i \vec{g}_i$ is any vector depending only on θ_1 and θ_2, we define $u^i|_\alpha$ and $u_i|_\alpha$ by the relations

$$(u^i \vec{g}_i)_{,\alpha} = u^i|_\alpha \vec{g}_i = u_i|_\alpha \vec{g}^i .$$

Then

$$(3.12) \qquad u^i|_\alpha = u^i{}_{,\alpha} + \Gamma^i_{j\alpha} u^j , \qquad u_i|_\alpha = u_{i,\alpha} - \Gamma^j_{i\alpha} u_j ,$$

$$u^i|_\alpha = g^{ij} u_j|_\alpha .$$

We observe that $u^i|_\alpha$ is not an ordinary covariant derivative since the $\Gamma^i_{j\alpha}$'s are not all Christoffel symbols and the u^i's are not the components of a tensor.

For doubly indexed sets $D^{i\alpha}$ depending only on θ_1 and θ_2 where the first index can have the values 1, 2, 3 and the second index the values 1, 2, we find it convenient to make the following definitions.

$$(3.13) \qquad \begin{cases} D^{i\alpha}|_\beta = D^{i\alpha}{}_{,\beta} + \Gamma^i_{j\beta} D^{j\alpha} + \Gamma^\alpha_{\gamma\beta} D^{i\gamma} \\ D_i{}^\alpha|_\beta = D_i{}^\alpha{}_{,\beta} - \Gamma^j_{i\beta} D_j{}^\alpha + \Gamma^\alpha_{\gamma\beta} D_i{}^\gamma \\ D^i{}_\alpha|_\beta = D^i{}_{\alpha,\beta} + \Gamma^i_{j\beta} D^j{}_\alpha - \Gamma^\gamma_{\alpha\beta} D^i{}_\gamma \\ D_{i\alpha}|_\beta = D_{i\alpha,\beta} - \Gamma^j_{i\beta} D_{j\alpha} - \Gamma^\gamma_{\alpha\beta} D_{i\gamma} \end{cases}$$

Again these symbols are related by raising and lowering indices in the usual way although they are not ordinary covariant derivatives.

4. Derivation of the Shell Theory

Let $h_1(\theta_1,\theta_2)$ and $h_2(\theta_1,\theta_2)$ be negative and positive functions respectively. For the undeformed shell we take the region $h_1 \leq \theta_3 \leq h_2$. Thus the shell can have variable thickness, and, although we shall continue to call the surface with position vector $\vec{r}$ the undeformed middle surface, we see that it need not be a middle surface in the usual sense.

Let $\vec{\bar{r}} = \vec{\bar{r}}(\theta_1,\theta_2)$ be the position vector to the surface into which the undeformed middle surface goes under the deformation. This surface is called the deformed middle surface, and it is parameterized by θ_1 and θ_2 in such a way that each particle has the same parameter values on both surfaces.

We let v^i be the components of $\vec{\bar{r}}$ with respect to the $\vec{g}_i$'s. Then

$$(4.1) \qquad \vec{\bar{r}} = v^i\vec{g}_i = v_i\vec{g}^i \quad .$$

Letting $\vec{\bar{g}}_\alpha = \vec{\bar{r}}_{,\alpha}$ we have

$$(4.2) \qquad \vec{\bar{g}}_\alpha = v^i|_\alpha\vec{g}_i = v_i|_\alpha\vec{g}^i \quad .$$

Next let $\vec{\bar{g}}_3 = \dfrac{\vec{\bar{g}}_1 \times \vec{\bar{g}}_2}{|\vec{\bar{g}}_1 \times \vec{\bar{g}}_2|}$ and let w^i be the components of $\vec{\bar{g}}_3$ with respect to the $\vec{g}_i$'s. Then

$$\vec{g}_3 = w^i \vec{g}_i = w_i \vec{g}^i \quad . \tag{4.3}$$

The condition that $\vec{g}_3$ is a unit normal vector to the deformed middle surface yields the following:

$$w_i w^i = 1 \ , \quad w_i v^i|_\alpha = 0 \ , \quad w_i w^i|_\alpha = 0 \ . \tag{4.4}$$

In the following all barred quantities have the same relationship to the deformed middle surface that the unbarred quantities have to the undeformed middle surface. We will use such quantities without making further definitions. Indices of such quantities are raised and lowered using the quantities $\bar{g}^{ij}$ and $\bar{g}_{ij}$.

We consider deformations in which normals to the undeformed middle surface go into normals to the deformed middle surface with no change in length along normals. This condition becomes

$$\vec{\bar{x}} = (v^i + \theta_3 w^i)\vec{g}_i \quad . \tag{4.5}$$

Next we calculate P_{ij}. We have

$$\frac{\partial \vec{\bar{x}}}{\partial x_j} = \frac{\partial \theta_\ell}{\partial x_j} \frac{\partial}{\partial \theta_\ell} [(v^k + \theta_3 w^k)\vec{g}_k] = \frac{\partial \theta_\alpha}{\partial x_j} (v^k|_\alpha + \theta_3 w^k|_\alpha)\vec{g}_k + \frac{\partial \theta_3}{\partial x_j} w^k \vec{g}_k \ .$$

Thus

$$p_{ij} = (v^k|_\alpha + \theta_3 w^k|_\alpha) \frac{\partial \theta_\alpha}{\partial x_j} a^i_k + w^k \frac{\partial \theta_3}{\partial x_j} a^i_k \ .$$

Hence

$$P_{ij} = a^m_i a^n_j p_{mn} = a^m_i a^n_j \left\{ (v^k|_\alpha + \theta_3 w^k|_\alpha) \frac{\partial\theta_\alpha}{\partial x_n} a^m_k + w^k \frac{\partial\theta_3}{\partial x_n} a^m_k \right\}$$

$$= g_{ik} a^n_j \left\{ (v^k|_\alpha + \theta_3 w^k|_\alpha) \frac{\partial\theta_\alpha}{\partial x_n} + w^k \frac{\partial\theta_3}{\partial x_n} \right\}$$

$$= a^n_j \left\{ (v_i|_\alpha + \theta_3 w_i|_\alpha) \frac{\partial\theta_\alpha}{\partial x_n} + w_i \frac{\partial\theta_3}{\partial x_n} \right\} .$$

To complete the derivation we compute $\frac{\partial\theta_i}{\partial x_j}$.

$$d\vec{x} = (\vec{r} + \theta_3 \vec{g}_3)_{,i} d\theta_i = \vec{g}_\alpha + \theta_3 \Gamma^k_{3\alpha} \vec{g}_k) d\theta_\alpha + \vec{g}_3 d\theta_3$$

$$= (\delta^\beta_\alpha - \theta_3 b^\beta_\alpha) \vec{g}_\beta d\theta_\alpha + \vec{g}_3 d\theta_3 .$$

Thus

$$\vec{g}^\gamma \cdot d\vec{x} = (\delta^\gamma_\alpha - \theta_3 b^\gamma_\alpha) d\theta_\alpha \quad \text{and} \quad \vec{g}^3 \cdot d\vec{x} = d\theta_3 .$$

Now let

$$(B^\alpha_\beta) = (\delta^\alpha_\beta - \theta_3 b^\alpha_\beta)^{-1} . \tag{4.6}$$

Then

$$d\theta_\alpha = B^\alpha_\gamma \vec{g}^\gamma \cdot d\vec{x} = B^\alpha_\gamma A^\gamma_i dx_i \quad \text{and} \quad d\theta_3 = \vec{g}^3 \cdot d\vec{x} = A^3_i dx_i .$$

Hence

$$\frac{\partial\theta_\alpha}{\partial x_i} = B^\alpha_\gamma A^\gamma_i \quad \text{and} \quad \frac{\partial\theta_3}{\partial x_i} = A^3_i . \tag{4.7}$$

Using (4.7) in the above expression for P_{ij}, we obtain

$$P_{i\alpha} = (v_i|_\beta + \theta_3 w_i|_\beta) B^\beta_\alpha , \quad P_{i3} = w_i . \tag{4.8}$$

Let $z^i_j = p^{ki} P_{kj}$. Then from (4.8) and (4.4) we have

$$z^\alpha_3 = (v^k|_\beta + \theta_3 w^k|_\beta) B^{\beta\alpha} w_k = 0 , \quad z^3_\alpha = w^k (v_k|_\beta + \theta_3 w_k|_\beta) B^\beta_\alpha = 0 ,$$

and

$$z^3_3 = w^k w_k = 1 \ .$$

From this we see that the vector having components 0, 0, 1 with respect to the $\vec{g}_i$'s (namely $\vec{g}_3$) has a direction of principal strain in the undeformed shell. Since $\vec{g}_3$ goes into $\vec{\bar{g}}_3$ in the deformation, it follows that $\vec{\bar{g}}_3$ has a direction of principal strain in the deformed shell, and the rotation matrix c rotates $\vec{g}_3$ into $\vec{\bar{g}}_3$. In terms of the C^{ij}'s this means

$$(C^{ij}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

or

$$C^{i3} = w^i \ .$$

Since $C_{i3}C^i{}_\alpha = g_{3\alpha} = 0$, we also have $w_i C^i{}_\alpha = 0$. Hence

$$\text{(4.9)} \qquad w_i C^i{}_\alpha = 0 \quad \text{and} \quad C^{i3} = w^i \ .$$

Using (4.8-9) and (3.8) it follows that

$$\text{(4.10)} \qquad w_i Q^{i\alpha} = 0 \text{ and } Q^{i3} \text{ is proportional to } w^i \ .$$

Next we start the computation of the total strain energy $E = \iiint W(P_{ij},\theta)dv$ under arbitrary variations of the deformed middle surface, i.e. under arbitrary variations of the v_i's. We have

$$\dot{E} = \iiint Q^{ij}\dot{P}_{ij}dv = \iiint [Q^{i\alpha}(\dot{v}_i|_\beta + \theta_3 \dot{w}_i|_\beta)B^\beta_\alpha + Q^{i3}\dot{w}_i]dv \ .$$

Since $w_i w^i = 1$, we have $\dot{w}_i w^i = 0$. Hence from (4.10), $Q^{i3}\dot{w}_i = 0$ and

$$\dot{E} = \iiint Q^{i\alpha}(\dot{v}_i|_\beta + \theta_3\dot{w}_i|_\beta)B^\beta_\alpha dv \ .$$

For dv we have

$$dv = \frac{\partial\vec{x}}{\partial\theta_1} \times \frac{\partial\vec{x}}{\partial\theta_2} \cdot \frac{\partial\vec{x}}{\partial\theta_3}\, d\theta_1 d\theta_2 d\theta_3 \ .$$

Since

$$\frac{\partial\vec{x}}{\partial\theta_\alpha} = \vec{g}_\alpha + \theta_3\Gamma^i_{3\alpha}\vec{g}_i = (\delta^\beta_\alpha - \theta_3 b^\beta_\alpha)\vec{g}_\beta \quad \text{and} \quad \frac{\partial\vec{x}}{\partial\theta_3} = \vec{g}_3 \ ,$$

this becomes

$$\begin{aligned} dv &= (\delta^\alpha_1 - \theta_3 b^\alpha_1)\vec{g}_\alpha \times (\delta^\beta_2 - \theta_3 b^\beta_2)\vec{g}_\beta \cdot \vec{g}_3 d\theta_1 d\theta_2 d\theta_3 \\ &= (\delta^\alpha_1 - \theta_3 b^\alpha_1)(\delta^\beta_2 - \theta_3 b^\beta_2)\varepsilon_{\alpha\beta 3}\vec{g}^3 \cdot \vec{g}_3 d\theta_1 d\theta_2 d\theta_3 \\ &= \sqrt{g}\ [e_{\alpha\beta}\delta^\alpha_1\delta^\beta_2 - \theta_3 e_{\alpha\beta}(\delta^\alpha_1 b^\beta_2 + b^\alpha_1\delta^\beta_2) + \theta_3^2 e_{\alpha\beta} b^\alpha_1 b^\beta_2] d\theta_1 d\theta_2 d\theta_3 \\ &= \sqrt{g}\ (1 - 2\theta_3 H + \theta_3^2 K) d\theta_1 d\theta_2 d\theta_3 \end{aligned}$$

where $H = \frac{1}{2}\, b^\alpha_\alpha$ is the mean curvature and $K = \det\ (b^\alpha_\beta)$ is the Gaussian curvature of the undeformed middle surface. Letting $dA = \sqrt{g}\ d\theta_1 d\theta_2$ be the element of area of the undeformed middle surface, we have

$$dv = (1 - 2\theta_3 H + \theta_3^2 K) dA d\theta_3 \ .$$

Now define

$$(4.11) \quad \begin{cases} S^{i\alpha} = \displaystyle\int_{h_1}^{h_2} Q^{i\beta}B^\alpha_\beta(1 - 2\theta_3 H + \theta_3^2 K)d\theta_3 \\ M^{i\alpha} = \displaystyle\int_{h_1}^{h_2} \theta_3 Q^{i\beta}B^\alpha_\beta(1 - 2\theta_3 H + \theta_3^2 K)d\theta_3 \end{cases}$$

Then

$$\dot{E} = \iint (S^{i\alpha}\dot{v}_i|_{\alpha} + M^{i\alpha}\dot{w}_i|_{\alpha})dA$$

so that $\dot{E}$ is reduced to an integral over the undeformed middle surface.

In the following we use the fact that $\sqrt{g}_{,\alpha} = \sqrt{g}\,\Gamma^{\beta}_{\alpha\beta}$. Then

$$\begin{aligned}\iint S^{i\alpha}\dot{v}_i|_{\alpha}dA &= \iint \sqrt{g}\,S^{i\alpha}(\dot{v}_{i,\alpha} - \Gamma^{j}_{i\alpha}\dot{v}_j)d\theta_1 d\theta_2 \\ &= \iint [(\sqrt{g}\,S^{i\alpha}\dot{v}_i)_{,\alpha} - \sqrt{g}\,\Gamma^{\beta}_{\alpha\beta}\,S^{i\alpha}\dot{v}_i \\ &\qquad - \sqrt{g}\,S^{i\alpha}{}_{,\alpha}\dot{v}_i - \sqrt{g}\,\Gamma^{j}_{i\alpha}\,S^{i\alpha}\dot{v}_j]d\theta_1 d\theta_2 \\ &= \iint [(\sqrt{g}\,S^{i\alpha}\dot{v}_i)_{,\alpha} - \sqrt{g}\,S^{i\alpha}|_{\alpha}\dot{v}_i]d\theta_1 d\theta_2 .\end{aligned}$$

By the divergence theorem

$$\iint (\sqrt{g}\,S^{i\alpha}\dot{v}_i)_{,\alpha}d\theta_1 d\theta_2 = \int e_{\alpha\beta}\sqrt{g}\,S^{i\alpha}\dot{v}_i d\theta_{\beta} = \int \varepsilon_{\alpha\beta}S^{i\alpha}\dot{v}_i \frac{d\theta_{\beta}}{ds}\,ds$$

where the integration aroung the boundary is in the positive direction when the undeformed middle surface is oriented so that the positive side is the one to which $\vec{g}_3$ points, ds being arc length along the boundary of the undeformed middle surface.

Let

$$(4.12) \qquad \lambda^{\alpha} = \frac{d\theta_{\alpha}}{ds}, \quad n_{\alpha} = \varepsilon_{\alpha\beta}\lambda^{\beta}, \quad \lambda^{\alpha} = \varepsilon^{\beta\alpha}n_{\beta} .$$

Then $\lambda^{\alpha}\vec{g}_{\alpha}$ is a unit tangent vector to the undeformed boundary curve, and $n^{\alpha}\vec{g}_{\alpha}$ is a unit outer vector which is normal to the undeformed boundary curve and is in the tangent plane of the middle undeformed surface.

Thus

$$\iint (\sqrt{g}\, S^{i\alpha}\dot{v}_i)_{,\alpha} d\theta_1 d\theta_2 = \int S^{i\alpha} n_\alpha \dot{v}_i ds$$

and

$$\iint S^{i\alpha}\dot{v}_i|_\alpha dA = \int S^{i\alpha} n_\alpha \dot{v}_i ds - \iint S^{i\alpha}|_\alpha \dot{v}_i dA \ .$$

Repeating this procedure for $\iint M^{i\alpha}\dot{w}_i|_\alpha dA$ we obtain

$$\dot{E} = \int (S^{i\alpha}\dot{v}_i + M^{i\alpha}\dot{w}_i) n_\alpha ds - \iint (S^{i\alpha}|_\alpha \dot{v}_i + M^{i\alpha}|_\alpha \dot{w}_i) dA \ .$$

Next we observe that w_i is a function of the $v_j|_\alpha$'s, and we let

$$(4.13) \qquad w_i^{\ j\alpha} = \frac{\partial w_i}{\partial v_j|_\alpha} \ , \qquad w^{ij\alpha} = \frac{\partial w_i}{\partial v_j|_\alpha} \ , \qquad w^{ij\alpha} = g^{ik} w_k^{\ j\alpha} \ .$$

Then $\dot{w}_i = w_i^{\ j\alpha}\dot{v}_j|_\alpha$ and

$$\dot{E} = \int (S^{i\alpha} n_\alpha \dot{v}_i + M^{j\beta} n_\beta w_j^{\ i\alpha} \dot{v}_i|_\alpha) ds$$

$$- \iint (S^{i\alpha}|_\alpha \dot{v}_i + M^{j\beta}|_\beta w_j^{\ i\alpha} \dot{v}_i|_\alpha) dA \ .$$

Methods used above show that

$$\iint M^{j\beta}|_\beta w_j^{\ i\alpha} \dot{v}_i|_\alpha dA = \int M^{j\beta}|_\beta w_j^{\ i\alpha} n_\alpha \dot{v}_i ds$$

$$- \iint (M^{j\beta}|_\beta w_j^{\ i\alpha})|_\alpha \dot{v}_i dA \ .$$

Hence

$$(4.14) \qquad \dot{E} = \int [(S^{i\alpha} - M^{j\beta}|_\beta w_j^{\ i\alpha}) n_\alpha \dot{v}_i + M^{j\beta} n_\beta w_j^{\ i\alpha} \dot{v}_i|_\alpha] ds$$

$$- \iint (S^{i\alpha} - M^{j\beta}|_\beta w_j^{\ i\alpha})|_\alpha \dot{v}_i dA$$

which is the desired form for $\dot{E}$.

Let $T^i\vec{g}_i$ be the force vector applied to the edge of the shell. Then the principal of virtual work states that

$$\dot{E} = \iint T^i(\dot{v}_i + \theta_3\dot{w}_i)d\bar{S} = \iint T^i(\dot{v}_i + \theta_3 w_i{}^{j\alpha}\dot{v}_j|_\alpha)d\bar{S}$$

where $d\bar{S}$ is the element of area on the deformed edge.

Letting $\bar{s}$ be the arc length along the boundary curve of the deformed middle surface, we can write $d\bar{S}$ in the form $d\bar{S} = \bar{A}d\bar{s}d\theta_3$ where $\bar{A}$ depends on the v_i's and their first two derivatives. We define

$$(4.15)\qquad \begin{cases} F^i = \int_{h_1}^{h_2} T^i\bar{A}d\theta_3 \\ G^i = \int_{h_1}^{h_2} \theta_3 T^i\bar{A}d\theta_3 \end{cases}$$

Then $F^i\vec{g}_i$ and $G^i\vec{g}_i$ are called the force vector per unit deformed arc length and the surface moment vector per unit deformed arc length. The phrase "surface moment vector" is used since $G^i\vec{g}_i$ is a moment aroung the middle deformed surface. The phrase "moment vector" is reserved for the moment of the force vector around the origin. Then the principal of virtual work becomes

$$\dot{E} = \int (F^i\dot{v}_i + G^j w_j{}^{i\alpha}\dot{v}_i|_\alpha)d\bar{s} \; .$$

Comparing this with (4.14) we obtain the differential equations

$$(4.16)\qquad (S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha})|_\alpha = 0$$

and the boundary conditions

$$(4.17) \qquad \int \left\{ \left[F^i \frac{d\bar{s}}{ds} - (S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha}) n_\alpha \right] \dot{v}_i + \left(G^j \frac{d\bar{s}}{ds} - M^{j\beta} n_\beta \right) w_j{}^{i\alpha} \dot{v}_i|_\alpha \right\} ds = 0$$

for all variations $\dot{v}_i$.

Since $S^{i\alpha}$ and $M^{i\alpha}$ are functions of the v^i's and their first two derivatives, we see that in (4.16) we have three fourth order equations or a twelfth order system. We will show later that this system can be expressed as a tenth order system. Meanwhile we extract what information we can from (4.17).

Equation (4.17) has the form

$$\int (A^i \dot{v}_i + B^{i\alpha} \dot{v}_i|_\alpha) ds = 0$$

for arbitrary functions $\dot{v}_i$. Using the relation

$$\dot{v}_i|_\alpha = \dot{v}_i|_\beta \lambda^\beta \lambda_\alpha + \dot{v}_i|_\beta n^\beta n_\alpha ,$$

we have

$$0 = \int [A^i \dot{v}_i + B^{i\alpha}(\dot{v}_i|_\beta \lambda^\beta \lambda_\alpha + \dot{v}_i|_\beta n^\beta n_\alpha)] ds$$

$$= \int \left(A^i \dot{v}_i + B^{i\alpha} \lambda_\alpha \frac{\delta \dot{v}_i}{\delta s} + B^{i\alpha} n_\alpha \frac{\delta \dot{v}_i}{\delta n} \right) ds$$

where

$$(4.18) \qquad \frac{\delta \dot{v}_i}{\delta s} = \dot{v}_i|_\beta \lambda^\beta , \qquad \frac{\delta \dot{v}_i}{\delta n} = \dot{v}_i|_\beta n^\beta .$$

Then $\frac{\delta \dot{v}_i}{\delta n} \vec{g}^i$ is the derivative of $\dot{v}_i \vec{g}^i$ with respect to arc length in the direction of $n^\alpha \vec{g}_\alpha$ and $\frac{\delta \dot{v}_i}{\delta s} \vec{g}^i$ is $\frac{d}{ds} (\dot{v}_i \vec{g}^i)$. Hence

on the undeformed boundary curve the values of $\frac{\delta \dot{v}_i}{\delta n}$ are independent of the values of $\dot{v}_i$ and $\frac{\delta \dot{v}_i}{\delta s}$. It follows that $B^{i\alpha} n_\alpha = 0$ and

$$
\begin{aligned}
0 &= \int \left(A^i \dot{v}_i + B^{i\alpha} \lambda_\alpha \frac{\delta \dot{v}_i}{\delta s} \right) ds \\
&= \int \left[A^i \dot{v}_i + \frac{d}{ds} \left(B^{i\alpha} \lambda_\alpha \dot{v}_i \right) - \dot{v}_i \frac{\delta}{\delta s} \left(B^{i\alpha} \lambda_\alpha \right) \right] ds \\
&= \int \left[A^i - \frac{\delta}{\delta s} \left(B^{i\alpha} \lambda_\alpha \right) \right] \dot{v}_i \, ds
\end{aligned}
$$

where

$$
\frac{\delta}{\delta s} \left(B^{i\alpha} \lambda_\alpha \right) \vec{g}_i = \frac{d}{ds} \left(B^{i\alpha} \lambda_\alpha \vec{g}_i \right) .
$$

From this we have $A^i = \frac{\delta}{\delta s} (B^{i\alpha} \lambda_\alpha)$. Hence (4.17) becomes

$$
(4.19) \quad
\begin{cases}
F^i \dfrac{d\bar{s}}{ds} = \left(S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha} \right) n_\alpha + \dfrac{\delta}{\delta s} \left[\left(G^j \dfrac{d\bar{s}}{ds} - M^{j\beta} n_\beta \right) w_j{}^{i\alpha} \lambda_\alpha \right] \\
\left(G^j \dfrac{d\bar{s}}{ds} - M^{j\beta} n_\beta \right) w_j{}^{i\alpha} n_\alpha = 0
\end{cases}
$$

where

$$
\frac{\delta}{\delta s} \left[\left(G^j \frac{d\bar{s}}{ds} - M^{j\beta} n_\beta \right) w_j{}^{i\alpha} \lambda_\alpha \right] \vec{g}_i = \frac{d}{ds} \left[\left(G^j \frac{d\bar{s}}{ds} - M^{j\beta} n_\beta \right) w_j{}^{i\alpha} \lambda_\alpha \vec{g}_i \right] .
$$

Equations (4.19) express the F^i's and G^i's implicitly. Our next task is to obtain an explicit expression for them. To do this we derive an explicit expression for the $w^{ji\alpha}$'s and several other relationships.

The relationship

$$
\vec{g}_3 = \frac{\vec{g}_1 \times \vec{g}_2}{|\vec{g}_1 \times \vec{g}_2|} = \frac{1}{\sqrt{g}} v_k|_1 \vec{g}^k \times v_\ell|_2 \vec{g}^\ell = \frac{1}{\sqrt{g}} \varepsilon^{k\ell j} v_k|_1 v_\ell|_2 \vec{g}_j
$$

implies

(4.20) $$w^j = \frac{1}{\sqrt{\bar g}} \varepsilon^{jk\ell} v_k|_1 v_\ell|_2 \,.$$

Then

$$\bar g_{\gamma\omega} = v^k|_\gamma v_k|_\omega = g^{k\ell} v_\ell|_\gamma v_k|_\omega$$

and

$$\frac{\partial \bar g_{\gamma\omega}}{\partial v_i|_\alpha} = \delta_{\alpha\gamma} v^i|_\omega + \delta_{\alpha\omega} v^i|_\gamma \,.$$

Since $\bar g = e_{\gamma\omega}\bar g_{\gamma 1}\bar g_{\omega 2}$, we have

$$\frac{\partial \bar g}{\partial v_i|_\alpha} = e_{\gamma\omega}[(\delta_{\alpha\gamma} v^i|_1 + \delta_{\alpha 1} v^i|_\gamma)\bar g_{\omega 2} + \bar g_{\gamma 1}(\delta_{\alpha 2} v^i|_\omega + \delta_{\alpha\omega} v^i|_2)]$$

$$= e_{\alpha\omega}\bar g_{\omega 2} v^i|_1 + e_{\gamma\alpha}\bar g_{\gamma 1} v^i|_2 + e_{\gamma\omega}(\delta_{\alpha 1}\bar g_{\omega 2} v^i|_\gamma + \delta_{\alpha 2}\bar g_{\gamma 1} v^i|_\omega)$$

$$= 2e_{\alpha\omega} e_{\gamma\nu} \bar g_{\omega\nu} v^i|_\gamma$$

and

$$\frac{\partial}{\partial v_i|_\alpha}\left(\frac{1}{\sqrt{\bar g}}\right) = -\frac{1}{2\bar g^{3/2}} 2e_{\alpha\omega} e_{\gamma\nu} \bar g_{\omega\nu} v^i|_\gamma$$

$$= \frac{1}{\sqrt{\bar g}} \bar\varepsilon^{\alpha\omega}\, \bar\varepsilon^{\nu\gamma}\, \bar g_{\omega\nu} v^i|_\gamma \,.$$

Thus

$$w^{ji\alpha} = \frac{\partial}{\partial v_i|_\alpha}\left(\frac{1}{\sqrt{\bar g}} \varepsilon^{jk\ell} v_k|_1 v_\ell|_2\right)$$

$$= \frac{1}{\sqrt{\bar g}} \bar\varepsilon^{\alpha\omega}\, \bar\varepsilon^{\nu\gamma}\, \bar g_{\omega\nu} v^i|_\gamma \varepsilon^{jk\ell} v_k|_1 v_\ell|_2$$

$$+ \frac{1}{\sqrt{\bar g}} \varepsilon^{jk\ell}(\delta_{ki}\delta_{\alpha 1} v_\ell|_2 + v_k|_1 \delta_{i\ell}\delta_{\alpha 2})$$

$$= \bar\varepsilon^{\alpha\omega}\, \bar\varepsilon^{\nu\gamma}\, \bar g_{\omega\nu} v^i|_\gamma w^j + \frac{1}{\sqrt{g\bar g}} (e_{ji\ell}\delta_{\alpha 1} v_\ell|_2 + e_{jki}\delta_{\alpha 2} v_k|_1)$$

$$= \bar\varepsilon^{\alpha\omega}\, \bar\varepsilon_{\omega\gamma}\, \bar g^{\gamma\nu} v^i|_\nu w^j + \frac{1}{\sqrt{g\bar g}} e_{ji\ell} e_{\alpha\beta} v_\ell|_\beta$$

so that

$$w^{ji\alpha} = \varepsilon^{jik} v_k|_\beta \bar{\varepsilon}^{\alpha\beta} - \bar{g}^{\alpha\beta} v^i|_\beta w^j \ . \tag{4.21}$$

Next

$$\vec{\bar{g}}_\alpha \times \vec{\bar{g}}_\beta = v^i|_\alpha \vec{g}_i \times v^j|_\beta \vec{g}_j = \varepsilon_{ijk} v^i|_\alpha v^j|_\beta \vec{g}^k \ .$$

On the other hand

$$\vec{\bar{g}}_\alpha \times \vec{\bar{g}}_\beta = \bar{\varepsilon}_{\alpha\beta 3} \vec{\bar{g}}^3 = \bar{\varepsilon}_{\alpha\beta} w_k \vec{g}^k \ .$$

Thus

$$\varepsilon^{ijk} v_i|_\alpha v_j|_\beta = \bar{\varepsilon}_{\alpha\beta} w^k \ .$$

Similarly

$$\vec{\bar{g}}_\alpha \times \vec{\bar{g}}_3 = v^i|_\alpha \vec{g}_i \times w^j \vec{g}_j = \varepsilon_{ijk} v^i|_\alpha w^j \vec{g}^k$$

and

$$\vec{\bar{g}}_\alpha \times \vec{\bar{g}}_3 = \bar{\varepsilon}_{\alpha 3\beta} \vec{\bar{g}}^\beta = \bar{\varepsilon}_{\beta\alpha} \vec{\bar{g}}^\beta = \bar{\varepsilon}^\beta{}_\alpha \vec{\bar{g}}_\beta = \bar{\varepsilon}^\beta{}_\alpha v_k|_\beta \vec{g}^k \ .$$

Thus

$$\varepsilon^{ijk} v_i|_\alpha w_j = \bar{\varepsilon}^\beta{}_\alpha v^k|_\beta \ .$$

Using the first of these two results and (4.21) we obtain

$$w^{ji\alpha} v_j|_\beta = \varepsilon^{jik} v_k|_\gamma \bar{\varepsilon}^{\alpha\gamma} v_j|_\beta = -\bar{\varepsilon}_{\beta\gamma} w^i \, \bar{\varepsilon}^{\alpha\gamma} = -\delta^\alpha_\beta w^i \ .$$

Using the second of the above two results and (4.21) we obtain

$$\begin{aligned} w^{ji\alpha} w_j &= \bar{\varepsilon}^\gamma{}_\beta v^i|_\gamma \bar{\varepsilon}^{\alpha\beta} - \bar{g}^{\alpha\beta} v^i|_\beta \\ &= \bar{\varepsilon}_{\omega\beta} \bar{g}^{\omega\gamma} v^i|_\gamma \bar{\varepsilon}^{\alpha\beta} - \bar{g}^{\alpha\beta} v^i|_\beta \\ &= \bar{g}^{\alpha\gamma} v^i|_\gamma - \bar{g}^{\alpha\beta} v^i|_\beta = 0 \ . \end{aligned}$$

We list these as

$$(4.22)\quad \begin{cases} \varepsilon^{ijk} v_i|_\alpha v_j|_\beta = \bar{\varepsilon}_{\alpha\beta} w^k \\ \varepsilon^{ijk} v_i|_\alpha w_j = \bar{\varepsilon}^{\beta}{}_{\alpha} v^k|_\beta \\ w^{ji\alpha} v_j|_\alpha = -\delta^\alpha_\beta w^i \\ w^{ji\alpha} w_j = 0 \end{cases}$$

Observe that $v^i|_\alpha \bar{\lambda}^\alpha$, $v^i|_\alpha \bar{n}^\alpha$, and w^i are the components with respect to the $\vec{g}_i$'s of three orthogonal vectors. In particular $v^i|_\alpha \bar{\lambda}^\alpha \bar{g}_i = \bar{\lambda}^\alpha \vec{\bar{g}}_\alpha$ is a unit tangent vector to the deformed boundary curve, $v^i|_\alpha \bar{n}^\alpha \vec{g}_i = \bar{n}^\alpha \vec{\bar{g}}_\alpha$ is a unit outer vector which is normal to the deformed boundary curve and is in the tangent plane to the deformed middle surface, and $w^i \bar{g}_i = \vec{\bar{g}}_3$ is a unit normal to the deformed middle surface. Hence there exist functions B, C, and D defined on the deformed boundary curve such that

$$(4.23A)\quad G_j \frac{d\bar{s}}{ds} - M_j{}^\beta n_\beta = B \frac{d\bar{s}}{ds} v_j|_\gamma \bar{\lambda}^\gamma + C v_j|_\gamma \bar{n}^\gamma + D \frac{d\bar{s}}{ds} w_j \, .$$

Using (4.22) we obtain

$$(4.23B)\quad \left(G_j \frac{d\bar{s}}{ds} - M_j{}^\beta n_\beta\right) w^{ji\alpha} = -\left(B \frac{d\bar{s}}{ds} \bar{\lambda}^\alpha + C\bar{n}^\alpha\right) w^i \, .$$

Since

$$\bar{\lambda}^\alpha = \frac{d\theta_\alpha}{d\bar{s}} = \frac{d\theta_\alpha}{ds} \frac{ds}{d\bar{s}} = \frac{ds}{d\bar{s}} \lambda^\alpha \, ,$$

we have

$$\bar{n}_\alpha = \bar{\varepsilon}_{\alpha\beta} \bar{\lambda}^\beta = \sqrt{\frac{\bar{g}}{g}} \, \varepsilon_{\alpha\beta} \lambda^\beta \frac{ds}{d\bar{s}} = \sqrt{\frac{\bar{g}}{g}} \frac{ds}{d\bar{s}} n_\alpha \, .$$

We list these as

$$(4.24) \qquad \bar{\lambda}^{\alpha} = \frac{ds}{d\bar{s}}\lambda^{\alpha} , \qquad \bar{n}_{\alpha} = \sqrt{\frac{\bar{g}}{g}}\frac{ds}{d\bar{s}} n_{\alpha} .$$

From (4.23B), (4.24), and (4.19) we have

$$(G_j \frac{d\bar{s}}{ds} - M_j{}^{\beta} n_{\beta}) w^{ji\alpha} n_{\alpha} = -(B \frac{d\bar{s}}{ds}\bar{\lambda}^{\alpha} + C\bar{n}^{\alpha}) w^i \sqrt{\frac{g}{\bar{g}}}\frac{d\bar{s}}{ds}\bar{n}_{\alpha}$$

$$= -C\sqrt{\frac{g}{\bar{g}}}\frac{d\bar{s}}{ds} w^i = 0 .$$

Therefore C = 0.

Next

$$(G_j \frac{d\bar{s}}{ds} - M_j{}^{\beta} n_{\beta}) w^{ji\alpha}\lambda_{\alpha} = -B\lambda^{\alpha} w^i \lambda_{\alpha} = -Bw^i .$$

Thus we can rewrite the first of equations (4.19) and equation (4.23A) as

$$(4.25) \qquad \begin{cases} F^i = (S^{i\alpha} - M^{j\beta}|_{\beta} w_j{}^{i\alpha}) n_{\alpha} \dfrac{ds}{d\bar{s}} - \dfrac{\delta}{\delta\bar{s}}(Bw^i) \\[2ex] G^i = M^{i\alpha} n_{\alpha} \dfrac{ds}{d\bar{s}} + Bv^i|_{\alpha}\bar{\lambda}^{\alpha} + Dw^i . \end{cases}$$

When the edge forces are prescribed, equations (4.25) provide only four constraints on the v^i's since the functions B and D are not determined by the principle of virtual work. Since in (4.16) we have a twelfth order system of equations, we are not yet in a position to say that the deformed surface is determined when the edge forces are prescribed. The first step in resolving this problem is to switch from differential equations and force-deformation relationships expressed with respect to the vectors $\vec{g}_i$ to those expressed in terms of the vectors $\vec{\bar{g}}_i$.

5. The Theory in Terms of the Deformed Middle Surface

Let $\bar{F}^i$ and $\bar{G}^i$ be the components of the force and surface moment vectors per unit deformed arc length with respect to the vectors $\vec{\bar{g}}_i$. Then for example $\bar{F}^i \vec{\bar{g}}_i = F^i \vec{g}_i$ so that

$$\bar{F}^\alpha v^i|_\alpha + \bar{F}^3 w^i = F^i \ .$$

Multiplying by $v_i|_\beta$ we obtain

$$\bar{F}^\alpha \bar{g}_{\alpha\beta} = \bar{F}_\beta = F^i v_i|_\beta \ .$$

Multiplying by w_i we obtain

$$\bar{F}^3 = F^i w_i \ .$$

Thus

$$(5.1) \qquad \begin{cases} \bar{F}_\alpha = F^i v_i|_\alpha \ , & \bar{G}_\alpha = G^i v_i|_\alpha \\ \bar{F}_3 = F^i w_i \ , & \bar{G}_3 = G^i w_i \end{cases}$$

Using (5.1), (4.24), and (4.25) we have

$$(5.2) \qquad \begin{cases} \bar{F}_\alpha = \sqrt{\dfrac{g}{\bar{g}}}\, v_i|_\alpha (S^{i\gamma} - M^{j\beta}|_\beta w_j{}^{i\gamma})\bar{n}_\gamma - v_i|_\alpha \dfrac{\delta}{\delta \bar{s}} (Bw^i) \\ \bar{F}_3 = \sqrt{\dfrac{g}{\bar{g}}}\, w_i (S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha})\bar{n}_\alpha - w_i \dfrac{\delta}{\delta \bar{s}} (Bw^i) \\ \bar{G}_\alpha = \sqrt{\dfrac{g}{\bar{g}}}\, v_i|_\alpha M^{i\beta} \bar{n}_\beta + B\bar{\lambda}_\alpha \\ \bar{G}_3 = \sqrt{\dfrac{g}{\bar{g}}}\, w_i M^{i\alpha} \bar{n}_\alpha + D \end{cases}$$

Equations (5.2) can be simplified considerably as follows. From (4.10) $w_i Q^{i\alpha} = 0$. Using this with (4.11) we have

$$w_i S^{i\alpha} = w_i M^{i\alpha} = 0 .$$

From (4.21) and (4.22)

$$w^{ji\alpha} w_i = \varepsilon^{jik} v_k|_\beta \bar{\varepsilon}^{\alpha\beta} w_i = -\bar{\varepsilon}^{\gamma}{}_{\beta} v^j|_\gamma \bar{\varepsilon}^{\alpha\beta}$$

$$= -\bar{g}^{\gamma\omega} \bar{\varepsilon}_{\omega\beta} v^j|_\gamma \bar{\varepsilon}^{\alpha\beta} = -\bar{g}^{\alpha\gamma} v^j|_\gamma$$

and

$$v_i|_\alpha w^{ji\beta} = v_i|_\alpha (\varepsilon^{jik} v_k|_\gamma \bar{\varepsilon}^{\beta\gamma} - \bar{g}^{\beta\gamma} v^i|_\gamma w^j)$$

$$= \bar{\varepsilon}_{\alpha\gamma} w^j \bar{\varepsilon}^{\beta\gamma} - \bar{g}^{\beta\gamma} \bar{g}_{\alpha\gamma} w^j = \delta^\beta_\alpha w^j - \delta^\beta_\alpha w^j = 0 .$$

Also

$$v_i|_\alpha \frac{\delta}{\delta \bar{s}} (B w^i) = \vec{\bar{g}}_\alpha \cdot \frac{d}{d\bar{s}} (B \vec{\bar{g}}_3) = \vec{\bar{g}}_\alpha \cdot \left(\frac{dB}{d\bar{s}} \vec{\bar{g}}_3 + B \bar{\Gamma}^{i}_{3\beta} \vec{\bar{g}}_i \bar{\lambda}^\beta\right)$$

$$= B \bar{\Gamma}^{\gamma}_{3\beta} \bar{g}_{\alpha\gamma} \bar{\lambda}^\beta = -B \bar{b}^\gamma_\beta \bar{g}_{\alpha\gamma} \bar{\lambda}^\beta = -B \bar{b}_{\alpha\beta} \bar{\lambda}^\beta ,$$

and

$$w_i \frac{\delta}{\delta \bar{s}} (B w^i) = \vec{\bar{g}}_3 \cdot \frac{d}{d\bar{s}} (B \vec{\bar{g}}_3) = \frac{dB}{d\bar{s}} .$$

With these results (5.2) becomes

$$(5.3) \quad \begin{cases} \bar{F}_\alpha = \sqrt{\dfrac{g}{\bar{g}}}\, v_i|_\alpha S^{i\gamma} \bar{n}_\gamma + B \bar{b}_{\alpha\beta} \bar{\lambda}^\beta \\[2ex] \bar{F}_3 = \sqrt{\dfrac{g}{\bar{g}}}\, v_j|_\gamma M^{j\beta}|_\beta \bar{n}^\gamma - \dfrac{dB}{d\bar{s}} \\[2ex] \bar{G}_\alpha = \sqrt{\dfrac{g}{\bar{g}}}\, v_i|_\alpha M^{i\beta} \bar{n}_\beta + B \bar{\lambda}_\alpha \\[2ex] \bar{G}_3 = D \end{cases}$$

We now define

$$(5.4)\qquad \begin{cases} \tau_\alpha{}^\beta = \sqrt{\frac{g}{\bar{g}}}\, v_i|_\alpha S^{i\beta} \\ \tau_{3\alpha} = \sqrt{\frac{g}{\bar{g}}}\, v_j|_\alpha M^{j\beta}|_\beta \\ \mathfrak{M}_\alpha{}^\beta = \sqrt{\frac{g}{\bar{g}}}\, v_i|_\alpha M^{i\beta} \end{cases}$$

The indices on the τ's and $\mathfrak{M}$'s are to be raised and lowered using the quantities $\bar{g}^{ij}$ and $\bar{g}_{ij}$. Then (5.3) becomes

$$(5.5)\qquad \begin{cases} \bar{F}^\alpha = \tau^{\alpha\beta}\bar{n}_\beta + B\bar{b}^\alpha_\beta\,\bar{\lambda}^\beta \\ \bar{F}^3 = \tau^{3\alpha}\bar{n}_\alpha - \frac{dB}{d\bar{s}} \\ \bar{G}^\alpha = \mathfrak{M}^{\alpha\beta}\bar{n}_\beta + B\bar{\lambda}^\alpha \\ \bar{G}^3 = D \end{cases}$$

Equations (5.5) are the desired force-deformation relations in terms of the vectors $\vec{\bar{g}}_i$. We now start the calculation of the differential equations in the desired form.

Observe that

$$\vec{r}_{,\alpha\beta} = (v^i\vec{g}_i)_{,\alpha\beta} = (v^i|_\alpha\vec{g}_i)_{,\beta}$$

$$= v^i|_{\alpha,\beta}\vec{g}_i + v^i|_\alpha\Gamma^j_{i\beta}\,\vec{g}_j = (v^i|_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha}v^i|_\gamma)\vec{g}_i\ .$$

Here $v^i|_{\alpha\beta} = (v^i|_\alpha)|_\beta$ as defined in (3.13). Also

$$\vec{\bar{r}}_{,\alpha\beta} = \vec{\bar{g}}_{\alpha,\beta} = \bar{\Gamma}^j_{\alpha\beta}\,\vec{\bar{g}}_j = \bar{\Gamma}^\gamma_{\alpha\beta}\,\vec{\bar{g}}_\gamma + \bar{\Gamma}^3_{\alpha\beta}\,\vec{\bar{g}}_3 = (\bar{\Gamma}^\gamma_{\alpha\beta}v^i|_\gamma + \bar{b}_{\alpha\beta}w^i)\vec{g}_i.$$

Therefore

$$v^i|_{\alpha\beta} = (\bar\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta})v^i|_\gamma + \bar b_{\alpha\beta} w^i \, . \tag{5.6}$$

From (5.4) and $w_i M^{i\alpha} = 0$ we have

$$M^{i\alpha} = \sqrt{\frac{\bar g}{g}}\, v^i|_\beta \, \mathcal{M}^{\beta\alpha} \, . \tag{5.7}$$

We define $\tau^{i\alpha}\|_\beta$ as in (3.13) with $D^{i\alpha}$, $|$, and $\Gamma^j_{i\alpha}$ replaced by $\tau^{i\alpha}$, $\|$, and $\bar\Gamma^j_{i\alpha}$ respectively. Since $\mathcal{M}^{3\alpha}$ is not defined, we define $\mathcal{M}^{\alpha\beta}\|_\gamma$ to be an ordinary covariant derivative with respect to the metric of the deformed middle surface, i.e.

$$\mathcal{M}^{\alpha\beta}\|_\gamma = \mathcal{M}^{\alpha\beta}{}_{,\gamma} + \bar\Gamma^\alpha_{\omega\gamma}\mathcal{M}^{\omega\beta} + \bar\Gamma^\beta_{\omega\gamma}\mathcal{M}^{\alpha\omega} \, . \tag{5.8}$$

From (5.7) and (5.6) we have (recall $\sqrt{g}_{,\alpha} = \sqrt{g}\,\Gamma^\beta_{\alpha\beta}$)

$$\begin{aligned}
M^{i\alpha}|_\alpha &= \left(\sqrt{\frac{\bar g}{g}}\, v^i|_\beta \mathcal{M}^{\beta\alpha}\right)_{,\alpha} + \Gamma^i_{j\alpha}\sqrt{\frac{\bar g}{g}}\, v^j|_\beta \mathcal{M}^{\beta\alpha} + \Gamma^\alpha_{\gamma\alpha}\sqrt{\frac{\bar g}{g}}\, v^i|_\beta \mathcal{M}^{\beta\gamma} \\
&= \sqrt{\frac{\bar g}{g}}\Big\{(\bar\Gamma^\gamma_{\gamma\alpha} - \Gamma^\gamma_{\gamma\alpha})v^i|_\beta \mathcal{M}^{\beta\alpha} + v^i|_{\beta,\alpha}\mathcal{M}^{\beta\alpha} + v^i|_\beta \mathcal{M}^{\beta\alpha}{}_{,\alpha} \\
&\qquad\qquad + \Gamma^i_{j\alpha}v^j|_\beta \mathcal{M}^{\beta\alpha} + \Gamma^\alpha_{\gamma\alpha}v^i|_\beta \mathcal{M}^{\beta\gamma}\Big\} \\
&= \sqrt{\frac{\bar g}{g}}\Big\{v^i|_\beta(\mathcal{M}^{\beta\alpha}{}_{,\alpha} + \bar\Gamma^\alpha_{\gamma\alpha}\mathcal{M}^{\beta\gamma}) + \mathcal{M}^{\beta\alpha}(v^i|_{\beta,\alpha} + \Gamma^i_{j\alpha}v^j|_\beta)\Big\} \\
&= \sqrt{\frac{\bar g}{g}}\Big\{v^i|_\beta(\mathcal{M}^{\beta\alpha}\|_\alpha - \bar\Gamma^\beta_{\gamma\alpha}\mathcal{M}^{\gamma\alpha}) + \mathcal{M}^{\beta\alpha}(v^i|_{\beta\alpha} + \Gamma^\gamma_{\beta\alpha}v^i|_\gamma)\Big\} \\
&= \sqrt{\frac{\bar g}{g}}\Big\{v^i|_\beta \mathcal{M}^{\beta\alpha}\|_\alpha + \bar b_{\beta\alpha}\mathcal{M}^{\beta\alpha} w^i\Big\} \, .
\end{aligned}$$

We list this as

(5.9) $$M^{i\alpha}|_\alpha = \sqrt{\frac{\bar{g}}{g}}\,(v^i|_\beta\, \mathcal{M}^{\beta\alpha}\|_\alpha + \bar{b}_{\beta\alpha}\,\mathcal{M}^{\beta\alpha}w^i) .$$

From (5.9) and (5.4) we have $\tau_{3\alpha} = \bar{g}_{\alpha\beta}\mathcal{M}^{\beta\gamma}\|_\gamma$ or $\tau^{3\alpha} = \mathcal{M}^{\alpha\beta}\|_\beta$, and from (4.22) and (5.9) we have

$$M^{j\beta}|_\beta w_j{}^{i\alpha} = -\sqrt{\frac{\bar{g}}{g}}\,\mathcal{M}^{\alpha\beta}\|_\beta\, w^i = -\sqrt{\frac{\bar{g}}{g}}\,\tau^{3\alpha}w^i .$$

We list these as

(5.10) $$\begin{cases} \tau^{3\alpha} = \mathcal{M}^{\alpha\beta}\|_\beta \\ M^{j\beta}|_\beta w_j{}^{i\alpha} = -\sqrt{\frac{\bar{g}}{g}}\,\tau^{3\alpha}w^i \end{cases}$$

From (5.4) and $w_i S^{i\alpha} = 0$ we obtain $S^{i\alpha} = \sqrt{\frac{\bar{g}}{g}}\, v^i|_\beta \tau^{\beta\alpha}$. Hence from (5.10)

$$S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha} = \sqrt{\frac{\bar{g}}{g}}\,(v^i|_\beta \tau^{\beta\alpha} + \tau^{3\alpha}w^i) .$$

Letting

$$D^{i\alpha} = S^{i\alpha} - M^{j\beta}|_\beta w_j{}^{i\alpha} = \sqrt{\frac{\bar{g}}{g}}\,(v^i|_\beta \tau^{\beta\alpha} + \tau^{3\alpha}w^i) ,$$

the differential equations (4.16) become $D^{i\alpha}|_\alpha = 0$. This means that

$$(\sqrt{g}\, D^{i\alpha}\vec{g}_i)_{,\alpha} = \sqrt{g}\,(\Gamma^{\beta}_{\alpha\beta}D^{i\alpha}\vec{g}_i + D^{i\alpha}{}_{,\alpha}\vec{g}_i + D^{i\alpha}\Gamma^{j}_{i\alpha}\vec{g}_j)$$

$$= \sqrt{g}\, D^{i\alpha}|_\alpha \vec{g}_i = 0 .$$

But

$$\sqrt{g}\, D^{i\alpha}\vec{g}_i = \sqrt{\bar{g}}\,(v^i|_\beta \tau^{\beta\alpha} + \tau^{3\alpha}w^i)\vec{g}_i$$

$$= \sqrt{\bar{g}}\,(\tau^{\beta\alpha}\vec{\bar{g}}_\beta + \tau^{3\alpha}\vec{\bar{g}}_3) = \sqrt{\bar{g}}\,\tau^{i\alpha}\vec{\bar{g}}_i .$$

Therefore

$$0 = (\sqrt{g}\,\tau^{i\alpha}\vec{\bar{g}}_i)_{,\alpha} = \sqrt{g}\,\tau^{i\alpha}\|_\alpha \vec{\bar{g}}_i$$

as was the case for $D^{i\alpha}$ and the differential equations associated with the vectors $\vec{\bar{g}}_i$ are

$$(5.11) \qquad \tau^{i\alpha}\|_\alpha = 0 \;.$$

Since $S^{i\alpha}$ and $M^{i\alpha}$ depend on v^i and its first two derivatives, we see from (5.4) that $\tau^{\alpha\beta}$ depends on v^i and its first two derivatives whereas $\tau^{3\alpha}$ depends on v^i and its first three derivatives. Since $\bar{\Gamma}^{\,j}_{i\alpha} = \vec{\bar{g}}^j \cdot \vec{\bar{g}}_{i,\alpha}$, we see that $\bar{\Gamma}^{\,j}_{i\alpha}$ depends on v^i and its first two derivatives. Therefore (5.11) is a system of two third order equations and one fourth order equation making (5.11) a tenth order system of equations. We expect then that we need five boundary conditions to determine the solution. After prescribing the edge forces (5.5) gives us four constraints. Prescribing the functions B or D would give one additional constraint determining the solution unless buckling occurs. We next look for a reasonable criterion for the choice of B or D. We will check first to see if the equilibrium laws for force and the moment of force give a criterion for such a choice.

6. The Equilibrium Laws

Let $\bar{T}^i\vec{\bar{g}}_i$ be the edge force vector per unit deformed area. Then the total edge force vector is

$$\iint \bar{T}^i\vec{\bar{g}}_i\,d\bar{S} = \int \bar{F}^i\vec{\bar{g}}_i\,d\bar{s} = \int \left(\tau^{i\alpha}\bar{n}_\alpha\vec{\bar{g}}_i + B\bar{b}^\alpha_\beta\bar{\lambda}^\beta\vec{\bar{g}}_\alpha - \frac{dB}{ds}\vec{\bar{g}}_3\right)d\bar{s} \;.$$

We have

$$\int \tau^{i\alpha}\bar{n}_\alpha \vec{\bar{g}}_i d\bar{s} = \int \sqrt{\bar{g}}\ \tau^{i\alpha}\vec{\bar{g}}_i e_{\alpha\beta}\bar{\lambda}^\beta d\bar{s}$$

$$= \int \sqrt{\bar{g}}\ \tau^{i\alpha}\vec{\bar{g}}_i e_{\alpha\beta} d\theta_\beta = \iint (\sqrt{\bar{g}}\ \tau^{i\alpha}\vec{\bar{g}}_i)_{,\alpha} d\theta_1 d\theta_2$$

$$= \iint \sqrt{\bar{g}}\ \tau^{i\alpha}\|_\alpha \vec{\bar{g}}_i d\theta_1 d\theta_2 = 0 \ .$$

Also

$$\int (B\bar{b}^\alpha_\beta \bar{\lambda}^\beta \vec{\bar{g}}_\alpha - \frac{dB}{d\bar{s}} \vec{\bar{g}}_3)d\bar{s} = \int (-B\bar{\Gamma}^{\ \alpha}_{3\beta} \bar{\lambda}^\beta \vec{\bar{g}}_\alpha - \frac{dB}{d\bar{s}} \vec{\bar{g}}_3)d\bar{s}$$

$$= - \int (B\vec{\bar{g}}_{3,\beta}\bar{\lambda}^\beta + \frac{dB}{d\bar{s}} \vec{\bar{g}}_3)d\bar{s}$$

$$= - \int \frac{d}{d\bar{s}} (B\vec{\bar{g}}_3)d\bar{s} = 0 \ .$$

Thus the total force vector is zero regardless of the choice of B.

The total moment of force vector is

$$\iint \bar{T}^i\vec{\bar{g}}_i \times (\vec{\bar{r}} + \theta_3\vec{\bar{g}}_3)d\bar{S} = \int (\bar{F}^i\vec{\bar{g}}_i \times \vec{\bar{r}} + \bar{G}^i\vec{\bar{g}}_i \times \vec{\bar{g}}_3)d\bar{s}$$

$$= \int (\bar{F}^i\vec{\bar{g}}_i \times \vec{\bar{r}} + \bar{G}^\alpha\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta)d\bar{s}$$

$$= \int [(\tau^{i\gamma}\vec{\bar{g}}_i \times \vec{\bar{r}} + \mathfrak{m}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta)\bar{n}_\gamma + B\bar{b}^\alpha_\beta\bar{\lambda}^\beta\vec{\bar{g}}_\alpha \times \vec{\bar{r}}$$

$$- \frac{dB}{d\bar{s}} \vec{\bar{g}}_3 \times \vec{\bar{r}} + B\bar{\lambda}^\alpha\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta]d\bar{s} \ .$$

First

$$\int \tau^{i\gamma}\vec{\bar{g}}_i \times \vec{\bar{r}}\ \bar{n}_\gamma d\bar{s} = \int \sqrt{g}\ \tau^{i\gamma}\vec{\bar{g}}_i \times \vec{\bar{r}}\ e_{\gamma\omega}d\theta_\omega$$

$$= \iint (\sqrt{g}\ \tau^{i\gamma}\vec{\bar{g}}_i \times \vec{\bar{r}})_{,\gamma}d\theta_1 d\theta_2$$

$$= \iint (\sqrt{g}\ \tau^{i\gamma}\|_\gamma \vec{\bar{g}}_i \times \vec{\bar{r}} + \sqrt{g}\ \tau^{i\gamma}\vec{\bar{g}}_i \times \vec{\bar{g}}_\gamma)d\theta_1 d\theta_2$$

$$= \iint (\tau^{\alpha\gamma}\bar{\varepsilon}_{\alpha\gamma}\vec{\bar{g}}_3 + \tau^{3\gamma}\bar{\varepsilon}_{\gamma\beta}\vec{\bar{g}}^\beta)d\bar{A} \quad .$$

Next

$$\int \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta \bar{n}_\gamma d\bar{s} = \int \sqrt{g}\ \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta e_{\gamma\omega}d\theta_\omega$$

$$= \iint (\sqrt{g}\ \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta)_{,\gamma}d\theta_1 d\theta_2$$

$$= \iint \sqrt{g}\ (\bar{\Gamma}^{\omega}_{\gamma\omega}\mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta + \mathcal{M}^{\alpha\gamma}{}_{,\gamma}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta$$

$$+ \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha,\gamma}\vec{\bar{g}}^\beta - \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\bar{\Gamma}^{\beta}_{i\gamma}\vec{\bar{g}}^i)d\theta_1 d\theta_2$$

$$= \iint [(\mathcal{M}^{\alpha\gamma}{}_{,\gamma} + \bar{\Gamma}^{\gamma}_{\omega\gamma}\mathcal{M}^{\alpha\omega})\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta$$

$$+ (\bar{\varepsilon}_{\beta\alpha,\gamma} - \bar{\Gamma}^{\omega}_{\beta\gamma}\bar{\varepsilon}_{\omega\alpha})\mathcal{M}^{\alpha\gamma}\vec{\bar{g}}^\beta + \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\bar{b}^\beta_\gamma\vec{\bar{g}}_3]d\bar{A}$$

$$= \iint [(\mathcal{M}^{\alpha\gamma}\|_\gamma - \bar{\Gamma}^{\alpha}_{\omega\gamma}\mathcal{M}^{\omega\gamma})\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta$$

$$+ (\bar{\varepsilon}_{\beta\alpha}\|_\gamma + \bar{\Gamma}^{\omega}_{\alpha\gamma}\bar{\varepsilon}_{\beta\omega})\mathcal{M}^{\alpha\gamma}\bar{g}^\beta + \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\bar{b}^\beta_\gamma\vec{\bar{g}}_3]d\bar{A}$$

where $\bar{\varepsilon}_{\beta\alpha}\|_\gamma$ is an ordinary covariant derivative and $\bar{\varepsilon}_{\beta\alpha}\|_\gamma = 0$. The above becomes

$$\iint (\mathcal{M}^{\alpha\gamma}\|_\gamma\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta + \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\bar{b}^\beta_\gamma\vec{\bar{g}}_3)d\bar{A}$$

$$= \iint (\tau^{3\alpha}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^\beta + \mathcal{M}^{\alpha\gamma}\bar{\varepsilon}_{\beta\alpha}\bar{b}^\beta_\gamma\vec{\bar{g}}_3)d\bar{A} \quad .$$

Finally

$$\int (E\bar{b}^{\alpha}_{\beta}\bar{\lambda}^{\beta}\vec{\bar{g}}_{\alpha} \times \vec{\bar{r}} - \frac{dB}{d\bar{s}}\vec{\bar{g}}_3 \times \vec{\bar{r}} + B\bar{\lambda}^{\alpha}\bar{\varepsilon}_{\beta\alpha}\vec{\bar{g}}^{\beta})d\bar{s}$$

$$= \int (-B\bar{\Gamma}^{\alpha}_{3\beta}\bar{\lambda}^{\beta}\vec{\bar{g}}_{\alpha} \times \vec{\bar{r}} - \frac{dB}{d\bar{s}}\vec{\bar{g}}_3 \times \vec{\bar{r}} + B\bar{n}_{\beta}\vec{\bar{g}}^{\beta})d\bar{s}$$

$$= \int (-B\bar{\lambda}^{\beta}\vec{\bar{g}}_{3,\beta} \times \vec{\bar{r}} - \frac{dB}{d\bar{s}}\vec{\bar{g}}_3 \times \vec{\bar{r}} + B\bar{n}_{\beta}\vec{\bar{g}}^{\beta})d\bar{s}$$

$$= \int [-\frac{d}{d\bar{s}}(B\vec{\bar{g}}_3) \times \vec{\bar{r}} + B\bar{n}_{\beta}\vec{\bar{g}}^{\beta}]d\bar{s}$$

$$= \int (B\vec{\bar{g}}_3 \times \frac{d\vec{\bar{r}}}{d\bar{s}} + B\bar{n}_{\beta}\vec{\bar{g}}^{\beta})d\bar{s} = \int (B\vec{\bar{g}}_3 \times \vec{\bar{g}}_{\alpha}\bar{\lambda}^{\alpha} + B\bar{n}_{\beta}\vec{\bar{g}}^{\beta})d\bar{s}$$

$$= \int B[\bar{\varepsilon}_{\alpha\beta}\bar{\lambda}^{\alpha}\vec{\bar{g}}^{\beta} + \bar{n}_{\beta}\vec{\bar{g}}^{\beta}]d\bar{s} = \int B(-\bar{n}_{\beta}\vec{\bar{g}}^{\beta} + \bar{n}_{\beta}\vec{\bar{g}}^{\beta})d\bar{s} = 0 .$$

The total moment of force vector is thus $\iint (\tau^{\alpha\beta} + \bar{b}^{\alpha}_{\gamma}\mathcal{M}^{\beta\gamma})\bar{\varepsilon}_{\alpha\beta}\vec{\bar{g}}_3 d\bar{A}$. Since this is independent of B and D, we see that this will not yield a criterion for choosing B or D. We continue, however, since we are interested in the equilibrium laws for their own sake. We see that the total moment of force vector is zero if $(\tau^{\alpha\beta} + \bar{b}^{\alpha}_{\gamma}\mathcal{M}^{\beta\gamma})$ is symmetric. We proceed to show that this is the case.

Since $c^*p = \sqrt{p^*p}$, then $c = p(\sqrt{p^*p})^{-1}$. Hence $C^{ij} = P^i{}_k D^{kj}$ where (D^{ij}) is symmetric. Then from (3.8) we see that $Q^{ij} = P^i{}_k \mathcal{Q}^{kj}$ where $(\mathcal{Q}^{kj})$ is symmetric. Since $w_i Q^{i\alpha} = 0$, $w_i P^i{}_{\alpha} = 0$, and $w_i P^i{}_3 = 1$, then $0 = w_i Q^{i\alpha} = w_i P^i{}_k \mathcal{Q}^{k\alpha} = \mathcal{Q}^{3\alpha}$. Hence $Q^{i\alpha} = P^i{}_{\beta}\mathcal{Q}^{\beta\alpha}$ where $(\mathcal{Q}^{\beta\alpha})$ is symmetric.

Since $\vec{\bar{g}}_{3,\alpha} = w^i|_{\alpha}\vec{g}_i$ and $\vec{\bar{g}}_{3,\alpha} = \bar{\Gamma}^{i}_{3\alpha}\vec{\bar{g}}_i = -\bar{b}^{\beta}_{\alpha}\vec{\bar{g}}_{\beta} = -\bar{b}^{\beta}_{\alpha}v^i|_{\beta}\vec{g}_i$, then $w^i|_{\alpha} = -\bar{b}^{\beta}_{\alpha}v^i|_{\beta}$. Hence from (4.8) we can write $P^i{}_{\alpha} = v^i|_{\beta}(\delta^{\beta}_{\gamma} - \theta_3\bar{b}^{\beta}_{\gamma})B^{\gamma}_{\alpha}$. Thus

$$\begin{aligned}\tau^{\alpha\beta} &= \bar{g}^{\alpha\gamma}\tau_{\gamma}{}^{\beta} = \bar{g}^{\alpha\gamma}\sqrt{\frac{g}{\bar{g}}}\, v_i|_{\gamma} S^{i\beta} \\ &= \sqrt{\frac{g}{\bar{g}}}\, \bar{g}^{\alpha\gamma} v_i|_{\gamma} \int Q^{i\omega} B^{\beta}_{\omega}(1 - 2H\theta_3 + K\theta_3^2)\,d\theta_3 \\ &= \sqrt{\frac{g}{\bar{g}}}\, \bar{g}^{\alpha\gamma} v_i|_{\gamma} \int v^i|_{\nu}(\delta^{\nu}_{\rho} - \theta_3\bar{b}^{\nu}_{\rho})B^{\rho}_{\lambda}\, \mathcal{Q}^{\lambda\omega} B^{\beta}_{\omega}(1 - 2H\theta_3 + K\theta_3^2)\,d\theta_3 \\ &= \sqrt{\frac{g}{\bar{g}}} \int (\delta^{\alpha}_{\rho} - \theta_3\bar{b}^{\alpha}_{\rho})B^{\rho}_{\lambda}\, \mathcal{Q}^{\lambda\omega} B^{\beta}_{\omega}(1 - 2H\theta_3 + K\theta_3^2)\,d\theta_3 \ .\end{aligned}$$

Let

$$D^{\alpha\beta} = \sqrt{\frac{g}{\bar{g}}}\, B^{\alpha}_{\gamma}\, \mathcal{Q}^{\gamma\omega} B^{\beta}_{\omega}(1 - 2H\theta_3 + K\theta_3^2) \ .$$

Then $(D^{\alpha\beta})$ is symmetric and

$$\tau^{\alpha\beta} = \int (\delta^{\alpha}_{\gamma} - \theta_3\bar{b}^{\alpha}_{\gamma})D^{\gamma\beta}\,d\theta_3 \ .$$

Similarly

$$\mathcal{M}^{\alpha\beta} = \int \theta_3(\delta^{\alpha}_{\gamma} - \theta_3\bar{b}^{\alpha}_{\gamma})D^{\gamma\beta}\,d\theta_3 \ .$$

Hence

$$\begin{aligned}\tau^{\alpha\beta} + \bar{b}^{\alpha}_{\gamma}\mathcal{M}^{\beta\gamma} &= \int (D^{\alpha\beta} - \theta_3\bar{b}^{\alpha}_{\gamma}D^{\gamma\beta} + \theta_3\bar{b}^{\alpha}_{\gamma}D^{\beta\gamma} - \theta_3^2\bar{b}^{\alpha}_{\gamma}\bar{b}^{\beta}_{\omega}D^{\omega\gamma})\,d\theta_3 \\ &= \int (D^{\alpha\beta} - \theta_3^2\bar{b}^{\alpha}_{\gamma}D^{\gamma\omega}\bar{b}^{\beta}_{\omega})\,d\theta_3 \ .\end{aligned}$$

Since $(D^{\alpha\beta})$ and $(\bar{b}^{\alpha}_{\gamma}D^{\gamma\omega}\bar{b}^{\beta}_{\omega})$ are symmetric, the desired conclusion follows. Hence the total moment of force vector is zero and the usual equilibrium laws are satisfied.

7. Choice of B

We finally obtain a reasonable criterion for choosing B by taking a close look at the parts of the force vector and surface moment vector per unit deformed arc length which do not depend on B or D. First from (5.4)

$$\tau_\alpha{}^\beta \bar{n}_\beta = \sqrt{\frac{g}{\bar{g}}}\, v_i|_\alpha S^{i\beta} \bar{n}_\beta = v_i|_\alpha S^{i\beta} n_\beta \frac{ds}{d\bar{s}}$$

$$= v_i|_\alpha \int Q^{i\gamma} B^\beta_\gamma b_\beta \frac{ds}{d\bar{s}} (1 - 2H\theta_3 + K\theta_3^2) d\theta_3 \ .$$

Let $\vec{N}$ be the unit outer normal vector and dS the element of area for the undeformed edge. Then

$$\vec{N} dS = \frac{d}{ds} (\vec{r} + \theta_3 \vec{g}_3) \times \frac{d}{d\theta_3} (\vec{r} + \theta_3 \vec{g}_3) ds d\theta_3$$

$$= (\delta^\beta_\alpha - \theta_3 b^\beta_\alpha) \lambda^\alpha \vec{g}_\beta \times \vec{g}_3 ds d\theta_3 = (\delta^\beta_\alpha - \theta_3 b^\beta_\alpha) \lambda^\alpha \varepsilon_{\gamma\beta} \vec{g}^\gamma \, ds d\theta_3 \ .$$

Let $N^\alpha \vec{g}_\alpha = \vec{N}$. Then

$$N_\gamma dS = (\delta^\beta_\alpha - \theta_3 b^\beta_\alpha) \lambda^\alpha \varepsilon_{\gamma\beta} ds d\theta_3 \ .$$

Multiplying by $\delta^\gamma_\omega - \theta_3 b^\gamma_\omega$ we obtain

$$N_\gamma (\delta^\gamma_\omega - \theta_3 b^\gamma_\omega) dS = \sqrt{g}\, e_{\gamma\beta} (\delta^\gamma_\omega - \theta_3 b^\gamma_\omega)(\delta^\beta_\alpha - {}_3 b^\beta_\alpha) \lambda^\alpha ds d\theta_3$$

$$= \sqrt{g}\, e_{\omega\alpha} \det (\delta^\nu_\mu - \theta_3 b^\nu_\mu) ds d\theta_3$$

$$= n_\omega (1 - 2H\theta_3 + K\theta_3^2) ds d\theta_3 \ .$$

Writing dS as $A ds d\theta_3$ we have

$$(7.1) \qquad N_\gamma(\delta^\gamma_\omega - \theta_3 b^\gamma_\omega)A = n_\omega(1 - 2H\theta_3 + K\theta_3^2) \; .$$

We now have

$$\tau_\alpha{}^\beta \bar{n}_\beta = v_i|_\alpha \int Q^{i\gamma} B^\beta_\gamma (\delta^\omega_\beta - \theta_3 b^\omega_3) N_\omega A \, \frac{ds}{d\bar{s}} \, d\theta_3$$

$$= v_i|_\alpha \int Q^{i\gamma} N_\gamma A \, \frac{ds}{d\bar{s}} \, d\theta_3 \; .$$

Recall from (2.2) that $q_{ij} n_j \dfrac{dS}{d\bar{S}}$ are the Cartesian coordinates for the edge force given by the three dimensional theory. In Section 2, (n_1, n_2, n_3) was the unit outer normal vector so the n_i's there are different from the n_α's. Hence

$$Q^{i\alpha} N_\alpha \frac{dS}{d\bar{S}} \vec{g}_i = Q^{i\alpha} N_\alpha \frac{A ds}{\bar{A} d\bar{s}} \vec{g}_i \qquad (d\bar{S} = \bar{A} d\bar{s} d\theta_3)$$

is the edge of force vector as computed by the three dimensional theory. Let $\bar{\mathcal{T}}_\alpha = v_i|_\alpha Q^{i\beta} N_\beta \dfrac{A ds}{\bar{A} d\bar{s}}$ and $\mathcal{T}_3 = w_i Q^{i\beta} N_\beta \dfrac{A ds}{\bar{A} d\bar{s}}$. Then $\bar{\mathcal{T}}^i \vec{g}_i$ is the edge force vector as computed from the three dimensional theory. Since $w_i Q^{i\alpha} = 0$, then $\bar{\mathcal{T}}^3 = 0$ and $\bar{\mathcal{T}}^i \vec{\bar{g}}_i = \bar{\mathcal{T}}^\alpha \vec{\bar{g}}_\alpha$ is tangent to the deformed middle surface.

Then $\tau_\alpha{}^\beta n_\beta = \int \bar{\mathcal{T}}_\alpha \bar{A} d\theta_3$. Hence $\tau^{\alpha\beta} n_\beta = \int \bar{\mathcal{T}}^\alpha \bar{A} d\theta_3$ are the components of force tangent to the deformed middle surface per unit deformed arc length as computed from the three dimensional theory. Hence if we choose $B = 0$ in (5.5), the stress strain laws of the three dimensional theory hold in the direction tangent to the deformed middle surface. We make that choice so that (5.5) becomes

$$(7.2)\qquad \begin{cases} \overline{F}^i = \tau^{i\alpha}\overline{n}_\alpha \\ \overline{G}^\alpha = \mathcal{M}^{\alpha\beta}\overline{n}_\beta \\ \overline{G}^3 = D \, . \end{cases}$$

Then it also follows as above that the surface moment vector as computed from the three dimensional theory is tangent to the deformed middle surface and is $\overline{G}^\alpha \vec{\overline{g}}_\alpha$. The component of force normal to the deformed middle surface is not consistent with the stress strain laws of the three dimensional theory, but it is such that the equilibrium laws are satisfied. The component of the surface moment vector normal to the deformed middle surface is also in general not consistent with the stress strain laws of the three dimensional theory. Its value is given algebraically in terms of the prescribed edge force and the deformation by $\overline{G}^3 = G^i w_i$.

8. Comparison with the von Karman-Föppl Equations

Consider a homogeneous isotropic plate which occupies the region $|x_3| \leq \frac{h}{2}$ in the undeformed state so that h is the thickness. Call $x_3 = 0$ the middle undeformed surface.

Choose $\theta_\alpha = x_\alpha$. Then $\theta_3 = x_3$ from (3.2) and $A^i_j = a^i_j = \delta^i_j$. Hence $P_{ij} = p_{ij}$, $C_{ij} = c_{ij}$, $Q_{ij} = q_{ij}$, and indices, which are raised and lowered with the g^{ij} and g_{ij}, have the same meaning when they are up as when they are down. We will write such indices in the lower position.

Since $\vec{g}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$, from (4.1), (4.3), and (3.10) we have $\vec{r} = (v_1, v_2, v_3)$, $\vec{\overline{g}}_3 = (w_1, w_2, w_3)$, and $\Gamma^j_{i\alpha} = 0$. Hence $v_i|_\alpha = v_{i,\alpha}$, etc.

The following are obtained from (4.11), (4.16), (4.20), and (4.21):

$$
(8.1) \quad \begin{cases}
S_{i\alpha} = \int_{-h/2}^{h/2} q_{i\alpha} dx_3 \,, \quad M_{i\alpha} = \int_{-h/2}^{h/2} x_3 q_{i\alpha} dx_3 \\
(S_{i\alpha} - M_{j\beta,\beta} w_{ji\alpha})_{,\alpha} = 0 \\
\overline{g}_{\alpha\beta} = v_{i,\alpha} v_{i,\beta} \\
w_i = \dfrac{1}{\sqrt{\overline{g}}} e_{ijk} v_{j,1} v_{k,2} \\
w_{ji\alpha} = \dfrac{1}{\sqrt{\overline{g}}} e_{jik} v_{k,\beta} e_{\alpha\beta} - \overline{g}^{\alpha\beta} v_{i,\beta} w_j
\end{cases}
$$

In the usual derivation of the von Karman-Föppl equations, in addition to the Kirchhoff hypothesis made already, it is assumed that $v_\alpha - x_\alpha$ and their derivatives are of order h^2 and v_3 and its derivatives are of order h. We incorporate these assumptions by assuming v_i is asymptotic to a power series in h,

$$v_i = \sum_{n=0}^{\infty} v_{in} h^n \,,$$

where $v_{\alpha o} = x_\alpha$, $0 = v_{\alpha 1} = v_{3o}$ and that differentiating these series term by term with respect to x_α yields a series which is asymptotic to the derivatives of v_i.

For convenience we let $v = v_{31}$. Then from (8.1)

$$(8.2)\quad \left\{\begin{aligned}
\bar{g}_{\alpha\beta} &= \delta_{\alpha\beta} + (v_{\alpha 2,\beta} + v_{\beta 2,\alpha} + v_{,\alpha} v_{,\beta}) h^2 + O(h^3) \\
(\bar{g}^{\alpha\beta}) &= (\bar{g}_{\alpha\beta})^{-1} = \delta_{\alpha\beta} - (v_{\alpha 2,\beta} + v_{\beta 2,\alpha} + v_{,\alpha} v_{,\beta}) h^2 + O(h^3) \\
\bar{g} &= 1 + (2 v_{\gamma 2,\gamma} + v_{,\gamma} v_{,\gamma}) h^2 + O(h^3) \\
\frac{1}{\sqrt{\bar{g}}} &= 1 - (v_{\gamma 2,\gamma} + \tfrac{1}{2} v_{,\gamma} v_{,\gamma}) h^2 + O(h^3) \\
w_\alpha &= -v_{,\alpha} h - v_{32,\alpha} h^2 + O(h^3) \\
w_3 &= 1 - \tfrac{1}{2} h^2 v_{,\gamma} v_{,\gamma} + O(h^3) \\
w_{\beta\gamma\alpha} &= O(h) \\
w_{3\beta\alpha} &= O(h^2) \\
w_{\beta 3\alpha} &= -\delta_{\alpha\beta} + O(h^2) \\
w_{33\alpha} &= O(h)
\end{aligned}\right.$$

From (4.8) $p_{i\alpha} = v_{i,\alpha} + x_3 w_{i,\alpha}$ and $p_{i3} = w_i$ so that

$$(8.3)\quad \left\{\begin{aligned}
p_{\alpha\beta} &= \delta_{\alpha\beta} - x_3 h v_{,\alpha\beta} + h^2 v_{\alpha 2,\beta} + O(h^3) \\
p_{3\alpha} &= v_{,\alpha} h + v_{32,\alpha} h^2 + O(h^3) \\
p_{\alpha 3} &= -v_{,\alpha} h - v_{32,\alpha} h^2 + O(h^3) \\
p_{33} &= 1 - \tfrac{1}{2} h^2 v_{,\gamma} v_{,\gamma} + O(h^3)
\end{aligned}\right.$$

Let $z_{ij} = p_{ki} p_{kj} - \delta_{ij}$. Then $z_{i3} = z_{3i} = 0$, and

$$(8.4)\quad z_{\alpha\beta} = O(h^2) = -2x_3hv_{,\alpha\beta} + h^2(v_{\alpha2,\beta} + v_{\beta2,\alpha} + v_{,\alpha}v_{,\beta}) + O(h^3) .$$

Let p, c, z, and q be the matrices (p_{ij}), (c_{ij}), (z_{ij}), and (q_{ij}) respectively. Then

$$(8.5)\quad \begin{cases} c^*p = \sqrt{p^*p} = \sqrt{1+z} = 1 + \frac{1}{2} z + O(h^4) \\ s_1 = [c^*p - 1] = \frac{1}{2}[z] + O(h^4) \\ s_2 = [(c^*p-1)^2] = O(h^4) , \quad s_3 = O(h^6) \\ c = p(\sqrt{p^*p})^{-1} = p[1 - \frac{1}{2} z + O(h^4)] \end{cases}$$

where the square bracket, [ ], denotes the trace of the matrix.

For a homogeneous isotropic material we want the strain energy density W to agree to lowest order terms with the strain energy density of the linear theory. That is, we want to have

$$W = \frac{\lambda}{2} s_1^2 + \mu s_2 + \text{higher order terms in } s_1, s_2, s_3$$

where λ and μ are the Lamé constants. It follows from (2.7) that

$$\begin{aligned} q &= W_1p + W_2c + W_3pc^*p \\ &= p\left\{W_1 + W_2[1 - \tfrac{1}{2} z + O(h^4)] + W_3[1 + \tfrac{1}{2} z + O(h^4)]\right\} \\ &= p\left\{\frac{\partial W}{\partial s_1} + \left(\frac{\partial W}{\partial s_2} - \frac{1}{2}\frac{\partial W}{\partial s_1}\right) z + O(h^4)\right\} \\ &= p\left\{\lambda s_1 + \mu z + O(h^4))\right\}. \end{aligned}$$

This with (8.1) gives

$$(8.6)\quad \begin{cases} q_{\alpha\beta} = \frac{\lambda}{2}\,[z]\delta_{\alpha\beta} + \mu z_{\alpha\beta} + O(h^4) \\ q_{3\alpha} = O(h^3) \\ S_{\alpha\beta} = h^3[\lambda\delta_{\alpha\beta}(v_{\gamma 2,\gamma} + \frac{1}{2}\, v_{,\gamma}v_{,\gamma}) \\ \qquad\qquad + \mu(v_{\alpha 2,\beta} + v_{\beta 2,\alpha} + v_{,\alpha}v_{,\beta})] + O(h^4) \\ M_{\alpha\beta} = -\frac{h^4}{12}\,(\lambda\delta_{\alpha\beta}\Delta v + 2\mu v_{,\alpha\beta}) + O(h^5) \\ S_{3\alpha} = O(h^4) \\ M_{3\alpha} = O(h^5) \end{cases}$$

where $\Delta v = v_{,\gamma\gamma}$.

Let $N_{i\alpha} = S_{i\alpha} - M_{j\beta,\beta}w_{ji\alpha}$. Then $N_{i\alpha,\alpha} = 0$ from (8.1). Defining $A_{\alpha\beta}$ as below, (8.6) and (8.2) yield

$$(8.7)\quad \begin{cases} A_{\alpha\beta} = \lambda\delta_{\alpha\beta}(v_{\gamma 2,\gamma} + \frac{1}{2}\, v_{,\gamma}v_{,\gamma}) + \mu(v_{\alpha 2,\beta} + v_{\beta 2,\alpha} + v_{,\alpha}v_{,\beta}) \\ N_{\alpha\beta} = S_{\alpha\beta} + O(h^5) = h^3A_{\alpha\beta} + O(h^4) \\ N_{3\alpha} = S_{3\alpha} - M_{\gamma\beta,\beta}w_{\gamma 3\alpha} + O(h^6) = S_{3\alpha} + M_{\alpha\beta,\beta} + O(h^6) \end{cases}$$

Hence $A_{\alpha\beta,\beta} = 0$ and since $A_{\alpha\beta} = A_{\beta\alpha}$, there is a function ϕ such that

$$(8.8)\quad \begin{cases} \phi_{,11} = A_{22} = \lambda(v_{12,1} + \frac{1}{2}\, v_{,1}v_{,1}) + (\lambda+2\mu)(v_{22,2} + \frac{1}{2}\, v_{,2}v_{,2}) \\ \phi_{,22} = A_{11} = (\lambda+2\mu)(v_{12,1} + \frac{1}{2}\, v_{,1}v_{,1}) + \lambda(v_{22,2} + \frac{1}{2}\, v_{,2}v_{,2}) \\ -\phi_{,12} = A_{12} = \mu(v_{12,2} + v_{22,1} + v_{,1}v_{,2}) \end{cases}$$

From the first two of (8.8)

$$v_{12,1} + \frac{1}{2} v_{,1}v_{,1} = \frac{(\lambda+2\mu)\phi_{,22} - \lambda\phi_{,11}}{4\mu(\lambda+\mu)}$$

$$v_{22,2} + \frac{1}{2} v_{,2}v_{,2} = \frac{(\lambda+2\mu)\phi_{,11} - \lambda\phi_{,22}}{4\mu(\lambda+\mu)} .$$

From these and the last of (8.8)

$$\begin{aligned}
-\phi_{,1122} &= A_{12,12} \\
&= \mu(v_{12,122} + v_{22,112} + v_{,112}v_{,2} + v_{,11}v_{,22} \\
&\qquad + v_{,12}v_{,12} + v_{,1}v_{,122}) \\
&= \mu\left[\frac{(\lambda+2\mu)\phi_{,2222} - \lambda\phi_{,1122}}{4\mu(\lambda+\mu)} + \frac{(\lambda+2\mu)\phi_{,1111} - \lambda\phi_{,1122}}{4\mu(\lambda+\mu)}\right. \\
&\qquad \left. - v_{,12}v_{,12} + v_{,11}v_{,22}\right] ,
\end{aligned}$$

$$\begin{aligned}
(8.9) \qquad \Delta^2\phi &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}(v_{,12}v_{,12} - v_{,11}v_{,22}) \\
&= \frac{E}{1-\sigma^2}(v_{,12}v_{,12} - v_{,11}v_{,22})
\end{aligned}$$

where E is Young's modulus, σ is Poisson's ratio and $\Delta^2\phi = \phi_{,\alpha\alpha\beta\beta}$.

From before (8.6) $q = p(\lambda s_1 + \mu z) + O(h^4)$ so that

$$\begin{aligned}
q_{3\alpha} &= hv_{,\beta}(\lambda s_1 \delta_{\beta\alpha} + \mu z_{\beta\alpha}) + O(h^4) \\
&= hv_{,\beta}\{h^2 A_{\alpha\beta} - x_3 h(\lambda\delta_{\alpha\beta}\Delta v + 2\mu v_{,\alpha\beta})\} + O(h^4) ,
\end{aligned}$$

and

$$S_{3\alpha} = h^4 A_{\alpha\beta} v_{,\beta} + O(h^5) .$$

From this and (8.7), (8.6)

$$N_{3\alpha} = h^4 A_{\alpha\beta} v_{,\beta} + M_{\alpha\beta,\beta} + O(h^5)$$

$$= h^4 \left\{ A_{\alpha\beta} v_{,\beta} - \frac{\lambda+2\mu}{12} (\Delta v)_{,\alpha} \right\} + O(h^5) ,$$

and

$$0 = N_{3\alpha,\alpha}$$

yields

$$(A_{\alpha\beta} v_{,\beta})_{,\alpha} - \frac{\lambda+2\mu}{12} \Delta^2 v = 0 ,$$

$$(8.10) \quad \Delta^2 v = \frac{12}{\lambda+2\mu} (\phi_{,11} v_{,22} + \phi_{,22} v_{,11} - 2\phi_{,12} v_{,12})$$

$$= \frac{12(1+\sigma)(1-2\sigma)}{E(1-\sigma)} (\phi_{,11} v_{,22} + \phi_{,22} v_{,11} - 2\phi_{,12} v_{,12}) .$$

Let

$$T_{\alpha\beta} = \left(\frac{1}{|p|} q_{\alpha i} p_{\beta i} \right) \Big|_{x_3=0} .$$

Then (see (2.3)) the $T_{\alpha\beta}$ are the components of the stress tensor on the middle surface as computed from the three dimensional theory. We obtain

$$T_{\alpha\beta} = q_{\alpha\beta} \Big|_{x_3=0} + O(h^3) = h^2 A_{\alpha\beta} + O(h^3) .$$

Let $\psi = h^2\phi$. Then ψ is the usual stress function satisfying $T_{11} = \psi_{,22} + O(h^3)$, $T_{22} = \psi_{,11} + O(h^3)$, $T_{12} = T_{21} = -\psi_{,12} + O(h^3)$.

Let $w = v_3$. Then $v = \frac{w}{h} + O(h)$ and (8.9) and (8.10) become

$$(8.11)\quad \begin{cases} \Delta^2\psi = \dfrac{E}{1-\sigma^2}\left(w_{,12}w_{,12} - w_{,11}w_{,22}\right) + O(h^3) \\[2ex] \Delta^2 w = \dfrac{12(1+\sigma)(1-2\sigma)}{h^2E(1-\sigma)}\left(\psi_{,11}w_{,22} + \psi_{,22}w_{,11} - 2\psi_{,12}w_{,12}\right) + O(h^2) \end{cases}$$

Equations (8.11) differ from the von Karman-Föppl equations in the constants $\dfrac{E}{1-\sigma^2}$ and $\dfrac{12(1+\sigma)(1-2\sigma)}{h^2E(1-\sigma)}$ which are E and $\dfrac{12(1-\sigma^2)}{h^2E}$ respectively for the von Karman-Föppl equations.

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